A comparison of mathematical modelling approaches for stability analysis of supply chains

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Abstract: Production and transportation processes along a supply chain are dynamic. In particular they are subject to perturbations (e.g., breakdown of a resource) that can destabilise the network. Stability is a major property of a supply chain that is essential for a sustainable relationship to its customers. In order to verify the stability of a given supply chain different criteria have been developed. This paper addresses the problem of choosing a proper mathematical modelling approach for a real world network in order to investigate stability. For this reason we discuss different modelling approaches. Each of these approaches can model different characteristics of a supply chain and features a specific stability criterion. By comparing these approaches the paper supports choosing a proper modelling approach for a real world supply chain.

Keywords: dynamics; modelling; stability; supply chain.

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1 Introduction

Supply chains often consist of production facilities around the world and serve local markets on different continents. Dynamic production and transportation processes along these chains have to be coordinated in order to create a sustainable network and to materialise its competitiveness (Christopher, 2005). A framework to diagnose collaborative supply chains and especially their information flows has been proposed by Lauras et al. (2009). Despite complex information and material flows a stochastic demand is challenging for supply chain managers. In particular capacity planning becomes a difficult task (Ettien et al., 2007). Planning and control of dynamic supply chains can be improved by precise knowledge about their behaviour. Stability is a major property of a given supply chain. In the case that a given network is stable it is able to meet the expectations of the customers in time, quantity and quality. Hence, the work in progress is bounded. This means for instance that a seasonally changing demand can be handled by the given resource capacities. On the other hand internal or external perturbations of production and transportation processes may lead to instability. For instance, a breakdown of a machine or transportation vehicle can be regarded as such an internal perturbation. Traffic jams are examples of external perturbations that increase the travel time between locations or customers. In the case of instability the work in progress grows unboundedly with time, high inventory cost for intermediate products occur and throughput times become large. Hence, the supply chain is not able to meet the customers demand. Stability criteria allow the evaluation of the mode of operation of a given supply chain. These criteria can be applied to a single location or to the whole network. Since the stability of all individual processes or locations is not sufficient for the stability of the network, the supply chain has to be considered as a single large-scale dynamical system. In the case of stability, robustness describes the kind and size of manageable perturbations before the system becomes unstable. Feature characteristics and dynamics of a supply chain can be modelled either by simulation models (Scholz-Reiter et al., 2005) or using mathematical modelling approaches. These approaches provide stability criteria or methods to investigate robustness. In the literature several modelling approaches have been developed. In Section 2 we introduce five of these. Each approach features a specific stability criterion and a method to investigate robustness of a real world supply chain. Section 3 compares the modelling approaches in regard to their capability to capture the properties of the real world network and applicable stability criteria. Some conclusions and suggestions for future research are presented in Section 4.

2 Mathematical modelling approaches for stability analysis

A supply chain typically consists of several locations, e.g., raw material suppliers, production facilities, warehouses and retailers. Each location can be considered as a single dynamical system that has to be modelled and analysed. The dynamics are for example given by time-varying production rates, available capacities or stock levels of incoming or finished products at a given location. An embedded location within the supply chain is connected to other locations by material, information and monetary flows. Such links are as well subject to time-varying capabilities, e.g., transportation capacity and speed. These links create an overall dynamical system. Hence, the dynamics of the supply chain are described by the dynamics of all single locations and their interconnections. In the following subsections we present five modelling approaches that capture the structure of a supply chain.

2.1 Damped oscillator models

A modelling approach inspired by physics of interconnected oscillators has been investigated in Helbing et al. (2004) and Helbing and Lämmer (2005). Here a supply chain is described as a physical transport problem, where the flows of products are considered. The model is given by balance equations for the flows of products and by the adaptation of the production speeds. There are n logistics locations denoted by $j \in \{1, \ldots, n\}$. Location j delivers d_{ij} products of kind i to other locations and consumes c_{kj} products of kind k per production cycle. The production speed $Q_j(t)$ of location j is the number of production cycles per time unit (day, week, ...). $N_i(t)$ denotes the number of products of kind i available in the supply chain (inventory). The function $Y_i(t)$ represents an external flow like consumption, losses minus the import of resources

$$Y_i(t) = c_{i,n+1}\, Q_{n+1}(t) - d_{i0}\, Q_0(t). \tag{1}$$

Here Q_{n+1} reflects the customers demand while Q_0 reflects the inflow of resources. It is assumed that c_{kj} and d_{ij} are normalised, such that $0 \leq c_{kj}, d_{ij} \leq 1$ and

$$d_{i0} = 1 - \sum_{j=1}^{n} d_{ij} \leq 0, \qquad c_{i,u+1} = 1 - \sum_{j=1}^{n} c_{ij} \leq 0. \tag{2}$$

The inventory change of product i is given by the difference of supply and demand

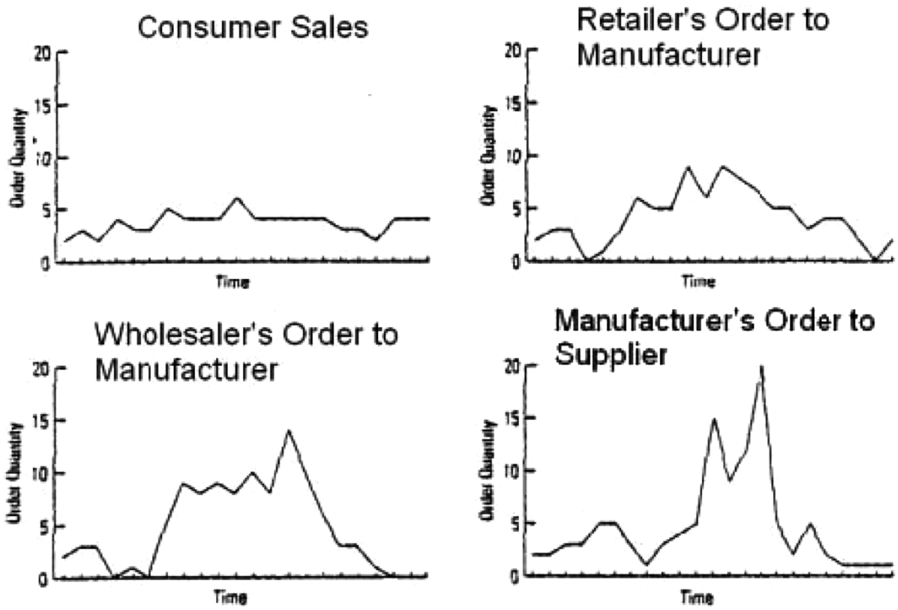
$$\frac{\mathrm{d}N_i}{\mathrm{d}t} = \sum_{j=1}^{n} d_{ij} Q_j(t) - \left[\sum_{j=1}^{n} c_{ij} Q_j(t) + Y_i(t) \right], \tag{3}$$

where the first term represents the supply and the second term denotes the demand. Variations of the consumption rate $Y_i(t)$ enforce an adaptation of production speeds. This is based on information about the current inventory of all locations i, the change of inventory $N_i(t)$ and the current production speed $Q_i(t)$. The adaptation is not instantaneous and requires an adaptation time T_i for adjustments. In the following we state an adapation for the case of sequential supply chains. Let $W_j(N_i, \frac{\mathrm{d}N_i}{\mathrm{d}t})$ denote a desired rate then the delivery rate is adapted according to

$$\frac{\mathrm{d}Q_i}{\mathrm{d}t} = \frac{1}{T_i}\left[W_j\left(N_i, \frac{\mathrm{d}N_i}{\mathrm{d}t}\right) - Q_i(t)\right]. \tag{4}$$

An instability occuring in such supply chains is the so-called bullwhip effect. The effect occurs in demand driven supply chains and was first studied by Forrester (1961). Different behavioural and operational effects lead to an increasing reaction of supply chain partners in upstream direction in regard to fluctuations of the final customer demand. A typical example of the bullwhip effect is shown in Figure 1 (Lee et al., 1997).

Figure 1 Increasing variability of orders up the supply chain



The analysis of the bullwhip effect is performed by linearising the model description (3), (4) around the equilibrium point $(N_i, \frac{\mathrm{d}N_i}{\mathrm{d}t}) = (\overline{N}_i, 0)$. The size of the bullwhip effect depends on the network topology and the adaptation of production speeds, see Helbing and Lämmer (2005). For the case of a sequential supply chain and feedback (4) the bullwhip effect occurs, if the adaptation time is too large. In the damped oscillator model the dynamics of a supply chain is represented by the flow of products and by the adaptation of the production rates. This approach provides qualitative models for the analytic analysis of the bullwhip effect, that may occur in a supply chain.

2.2 Decentralised supply chains

In decentralised (or autonomous) supply chains (Ouyang and Daganzo, 2006) information is not shared between all locations and each supplier determines its order quantities based on the demand and inventory information of previous time periods. A supply chain consists of $I+1$ suppliers connected sequentially. The suppliers are denoted by indices $i = 1, 2, \dots, I+1$ starting from downstream where $i = 0$ corresponds to the final customer. The time is discrete and the time periods are denoted by $t = 1, 2, \dots$ At the beginning of every time period t, supplier i checks his inventory level during the period and orders the needed quantity u_i at the end of the period. The inventory level of the ith supplier at the period $t+1$ is described as follows

$$x_i(t+1) = x_i(t) + u_i(t) - u_{i-1}(t), \qquad i = 1, 2, \dots, I. \tag{5}$$

Goods ordered by supplier i arrive after a constant lead time l_i. The in-stock inventory level of supplier i at the middle of the period $t+1$ is given by

$$y_i(t+1) = y_i(t) + u_i(t - l_i) - u_{i-1}(t), \qquad i = 1, 2, \dots, I. \tag{6}$$

The order quantity $u_i(t)$ of supplier i at the end of period t is calculated based on the information about its inventory levels x_i, y_i of all previous periods up to t and the order quantities u_{i-1} of all previous periods up to $t-1$. The next step is to focus on the ordering policy, which is based on the information above. Policies often used in practice are proper, Linear and Time-Invariant (LTI). A policy is called proper, if order sizes that are constant in time imply that

i the supplier inventory tends to a constant equilibrium value that is independent of the initial conditions

ii the orders placed tend to the value of orders received.

Further is a policy called LTI, if $u_i(t)$ is a time-dependent linear function of x_i, y_i and u_{i-1}. In order to give a simple description of a proper LTI policy we introduce the unit shift operator P for sequence and let P^k denote its k-fold application, i.e.,

$$P^k x_i(t) := x_i(t-k) \tag{7}$$

for all t and for all $k = 0, 1, \dots$ Then the general expression is

$$u_i(t) = \gamma_i + A_i(P)\, x_i(t) + B_i(P)\, y_i(t) + C_i(P)\, u_{i-1}(t-1), \qquad i = 1, 2, \dots, I. \tag{8}$$

Here γ_i is a real number and A_i, B_i and C_i are polynomials with real coefficients

$$A_i(P) = a_0^i + a_1^i\, P + a_2^i\, P^2 + \dots + a_n^i\, P^n, \tag{9}$$
$$B_i(P) = b_0^i + b_1^i\, P + b_2^i\, P^2 + \dots + b_n^i\, P^n, \tag{10}$$
$$C_i(P) = c_0^i + c_1^i\, P + c_2^i\, P^2 + \dots + c_n^i\, P^n, \tag{11}$$

where P is the shift operator. The polynomials A_i and B_i indicate the influence of inventory history on the ordering decisions and C_i the influence of orders received. If the order policy is proper, it can be shown that a nominal equilibrium exists such that order sizes, inventory levels and in-stock inventories stay constant, say x_i^∞, y_i^∞ and u_i^∞. To analyse the bullwhip effect we consider the error between the current states and their corresponding equilibrium. That is, we denote

$$\overline{x}_i(t) = x_i(t) - x_i^\infty, \quad \overline{y}_i(t) = y_i(t) - y_i^\infty, \quad \overline{u}_i(t) = u_i(t) - u_i^\infty \tag{12}$$

for all $i = 1, 2, \ldots, I$ and consider the ratio of the order sequences of the most upstream supplier and customer demand. This reflects the idea of the so-called *worst-case RMSE (root mean square error) amplification factor* (Ouyang and Daganzo, 2006) that is given by

$$W_I = \sup_{\overline{u}_0(\cdot)\neq 0} \left[\frac{\left(\sum_{t=0}^{\infty} \overline{u}_I^2(t)\right)^{\frac{1}{2}}}{\left(\sum_{t=0}^{\infty} \overline{u}_0^2(t)\right)^{\frac{1}{2}}} \right]. \tag{13}$$

This factor can be used to state whether supplier $I+1$ experiences a bullwhip effect or not. To be precise, in a supply chain, that is described within the error framework, supplier $I+1$ is said to experience no bullwhip effect if $W_I \leq 1$. In the case of proper LTI supply chains with $I+1$ suppliers (Ouyang and Daganzo, 2006) state that the condition

$$\sum_{i=1}^{I} \frac{1 + B_i(1)l_i - C_i(1)}{A_i(1) + B_i(1)} > 0 \tag{14}$$

is sufficient for the occurrence of the bullwhip effect. Furthermore there are similar analytical conditions for other policies (e.g., advanced demand information) to predict whether the bullwhip effect will occur or not, see Ouyang and Daganzo (2006). By using the transfer function, a standard technique from control theory, the amplification factor W_I is the H_∞-norm of the transfer function. So the usage of the transfer function forms a basis for the comparison of different policies with respect to instabilities.

2.3 Continuous dynamical systems

The following approach generalises the idea of the damped oscillator models. In this framework of continuous dynamical systems the approach to model the whole network begins with the modelling of the dynamics of each single location $i \in \{1, \ldots, N\}$ by a differential equation

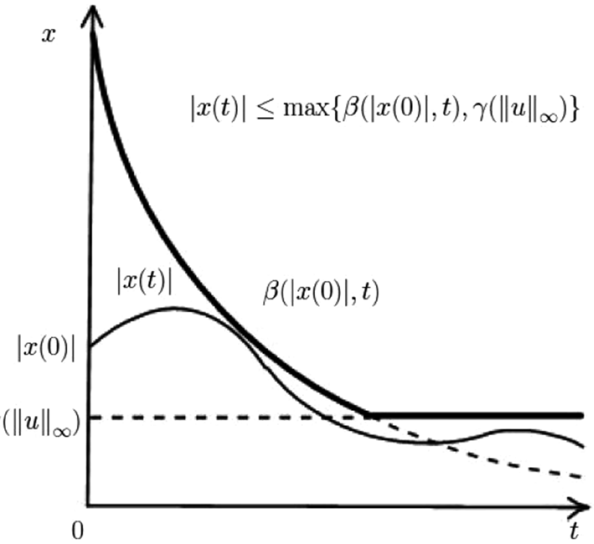
$$\dot{x}_i = f_i(x_1, \ldots, x_N, u_i), \tag{15}$$

where the functions need not to be linear. For instance, the state x_i describes the work in progress of location i. On the one hand the work in progress of a location i influences the work in progress of the other locations $j \neq i$. Besides that the state of a location i is also subject to an external input u_i. This input might be caused by new orders from customers of the supply chain. This allows to model the dependence and interconnections between the locations. In particular the supply of components and intermediate products can be modelled. The state of the whole supply chain is obtained by combining the states of all locations in one vector, i.e., $x = (x_1, \ldots, x_N)^T$. The dynamics of the supply chain is given by

$$\dot{x} = f(x, u) = \begin{pmatrix} f_1(x_1, \ldots, x_N, u_1) \\ \vdots \\ f_N(x_1, \ldots, x_N, u_N) \end{pmatrix}. \tag{16}$$

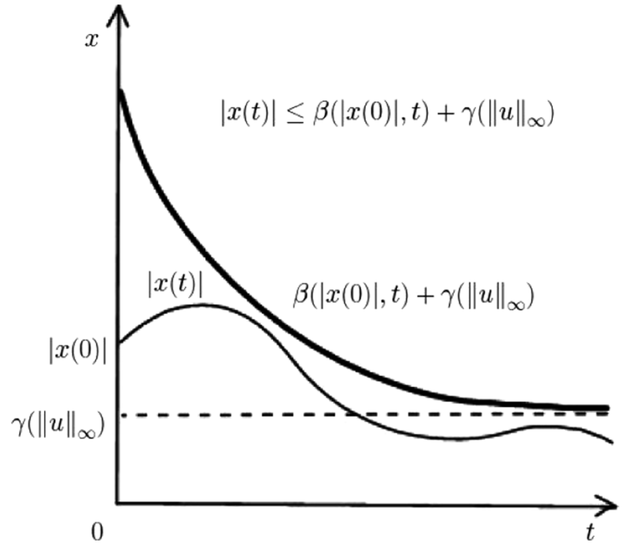
In mathematical systems theory a well established tool to analyse the stability of interconnected nonlinear dynamical systems is the notion of Input-to-State Stability (ISS). Figures 2 and 3 illustrate two equivalent ways of defining ISS. A precise definition can be found in Khalil (2002).

Figure 2 Trajectory of ISS continuous system



More concretely, in both figures the bold line represents the bounds on the state $x(t)$, e.g., the work in progress of a supply chain. Here it is shown that in the beginning the state $x(t)$ is bounded by some comparison function $\beta(|x(0)|, t)$ of the initial value of the system, which describes the overshoot resp. the transient phase and decays as time progresses. In the long time the influence of the initial value decreases and the state is bounded by a comparison function $\gamma(||u||_\infty)$, that depends on the input and reflects the maximal inflow in the interval of interest. A precise description of γ can be found in Khalil (2002). This framework may be used by considering the customers demand as external input and the state as the work in progress. In the case that the supply chain is ISS with respect to the customers demand, then the work in progress of the whole supply chain

Figure 3 Trajectory of ISS continuous system



remains bounded by the customers demand in the long run. This reflects the fact that an ISS supply chain is able to fulfill the customer demand. The notion of ISS is one posibility of defining stability, in which the external influences are addressed explicitly. A further advantage of the ISS notion is that there are stability criteria for interconnected systems (Dashkovskiy et al., 2007). For this reasoning this modelling approach allows for a modularity principle. That is, the ISS concept indicates how to establish a stable supply chain from stable single locations. Moreover, there are no restrictions on the interconnection structure. From the practical point of view this framework can cope with nonlinear dynamics in every location of the supply chain. Further, this approach provides stability criteria to decide whether the interconnection of stable locations leads to a stable supply chain. The criteria take the topology of the supply chain as well as the corresponding transportation processes into account.

2.4 Hybrid dynamical systems

The framework of hybrid dynamical systems is similar to that of continuous dynamical systems but the state of a supply chain is additionally allowed to be discontinuous in some time instants. Such discontinuities arise when there is an immediate change (jump) in the state of a location. This permits for instance a more detailed description of transportation processes. In particular, if the state represents the stock level then modelling discrete shippments of material, products etc. is possible. Moreover according to the state and the demand, that is denoted by u, a distinction of the kind of shipping can be drawn, e.g., shipping by truck, ship or airplane. The cases where the state, respectively the stock level, changes continuously are determined by a set C, i.e., the dynamics of location i is then given by

$$\dot{x}_i = f_i(x_1, \ldots, x_N, u_i), \qquad (x, u) \in C. \tag{17}$$

The discontinuities are described by the set D. The jumps in the state follow the equation

$$x_i^+ = g_i(x_1, \dots, x_N, u_i), \quad (x, u) \in D. \tag{18}$$

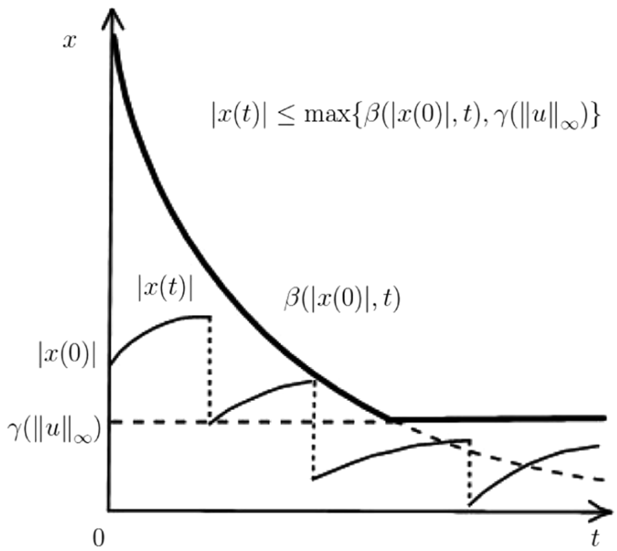
For a detailed description see Cai and Teel (2008). Analogously the dynamics of the supply chain is given by

$$\dot{x} = f(x, u), \quad (x, u) \in C, \tag{19}$$

$$x^+ = g(x, u), \quad (x, u) \in D. \tag{20}$$

The concept of ISS can be applied as well to analyse stability. The state is bounded by some functions of initial value $x(0)$ and the function of external input u. For a precise definition see Cai and Teel (2008). In the following Figures 4 and 5 the behaviour of a hybrid system that is ISS is shown.

Figure 4 Trajectory of ISS hybrid system

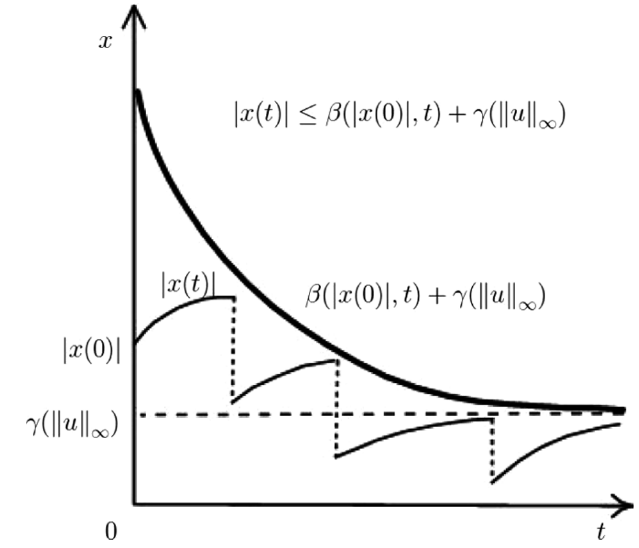


The bold lines bound the state of the supply chain, e.g., the stock level within the supply chain. The ISS concept for hybrid dynamical system also offers a small gain condition that guarantees the ISS property of the supply chain, if all single locations are ISS. This modelling approach has the same capabilities as mentioned in the previous subsection. In addition, this framework can map abrupt changes in the production strategy resp. policy. Also the abrupt ending or starting of material flow, information flow etc. can be modelled and thus transportation processes can be captured by the model in a more detailed manner.

2.5 Multiclass queueing networks and fluid approximation

Multiclass queueing networks are a well-established modelling approach to capture stochastic events that influence the discrete material flow of a supply chain (Dai and

Figure 5 Trajectory of ISS hybrid system



Jennings, 2003). Here only a brief description of a multiclass queueing network is given, for details see Dai (1995). The network consists of J locations that process K different types of products. The dynamics of the network can be described by the following stochastic processes. The arrival process $A_k(t)$ describes the number of external arrivals of type k products in the time period $[0, t]$. The production process $S_k(t)$ reflects the number of finished products of type k during the first t time units. For convenience we assume that each type of product is produced exclusively at one location. The mapping $s : \{1, \ldots, K\} \to \{1, \ldots J\}$ determines which type is produced at which location and generates the constituency matrix C, where $c_{jk} = 1$ if $s(k) = j$ and $c_{jk} = 0$ otherwise. After being processed products either change their type according to a given probability or leave the network. The routing process $R_k^l(n)$ denotes the number of type l products among the first n type l products that become products of type k. As each location can produce various product types a policy is needed that determines in which order the products are processed. Typical examples of such service disciplines are First-In-First-Out (FIFO), priority or processor sharing. The allocation process $T_k(t)$ denotes the total amount of time that location $s(k)$ has devoted on producing type k products. The initial amount of type k products is $Q_k(0)$ and the number of type k products at time t is given by the flow-balance equation

$$Q_k(t) = Q_k(0) + A_k(t) + \sum_{l=1}^{K} R_k^l(S_l(T_l(t))) - S_k(T_k(t)). \tag{21}$$

To obtain a complete description of the network dynamics further conditions on Q and T that depend on the service discipline have to be taken into account, see e.g., Chen and Zhang (1997, 2000). Roughly speaking a queueing network is said to be stable if the total number of products in the network remains bounded over all time. This can also be interpreted as saying that the long-run input rate of the network equals the long-run output rate. A precise definition for stability of multiclass queueing networks can be found in Bramson (2008) or Dai (1995). There the approach to the analysis of stability of multiclass queueing networks is to rescale the stochastic processes and to take limits (Dai, 1995). The so called fluid limit model is obtained by replacing the stochastic processes by their rates, i.e.,

$$\frac{1}{t}A_k(t) \to \alpha_k \qquad \frac{1}{t}S_k(t) \to \mu_k \qquad \frac{1}{t}R_k^l(t) \to p_{lk}. \tag{22}$$

The flow-balance equation in the continuous deterministic fluid model takes the form

$$Q_k(t) = Q_k(0) + \alpha_k(t) + \sum_{l=1}^{K} p_{lk}\mu_l T_l(t) - \mu_k T_k(t). \tag{23}$$

Again there are additional conditions on Q and T that are specific to the service discipline, see e.g., Chen and Zhang (1997, 2000). A fluid limit model is stable, if for all $k \in \{1, \ldots, K\}$ there is a finite time $\tau > 0$ such that for any $Q_k(\cdot)$ with $\sum_{k=1}^{K} Q_k(0) = 1$ it holds that $Q_k(\tau + \cdot) \equiv 0$. Dai (1995) has shown that the stability of the fluid limit model is sufficient for the stability of the multiclass queueing network. Consequently there is a purely deterministic criterion for the stability of a supply chain that is subject to stochastic uncertainties.

This modelling approach is suitable if the supply chain has highly reentrant flows. Further, there is huge variety of different service disciplines, which can be explicitly modelled in this framework. So simulations of different scenarios allow the choice of a policy that is suitable to the requirements of the supply chain. A further the strength of this approach is that analysis of the influence of stochastic uncertainties (e.g., production times, transportations etc.) on the stability is possible by purely deterministic criteria.

3 Comparison of the capabilities of the approaches

The presented modelling approaches vary in their capabilities. In regard to the dynamics continuous dynamical systems, hybrid dynamical systems and damped oscillator models are able to capture linear and nonlinear characteristics of logistics processes. Queueing networks, fluid models and decentralised supply chain models are able to describe linear dynamics. Furthermore continuous dynamical systems and hybrid dynamical systems can be used to describe the dynamics of a single location as well as the dynamics of the whole supply chain. In contrast to this queueing networks, damped oscillator models and decentralised supply chain models describe the dynamics of the whole network. Queueing networks as well as decentralised supply chain models (Ouyang and Daganzo, 2008) capture stochastic dynamics. The other presented approaches are completely deterministic. All modelling approaches have the ability to cover different types of products and to handle reentrant flows. Material and information flows between the locations determine the structure of a supply chain. Different intermediate and finished products circulate between the locations and form a comprehensive material flow. In order to model the structure of this flow all modelling approaches can be used to capture linear, convergent or divergent flows. Despite these basic properties the approaches differ in regard to their capabilities to model feature characteristics of a supply chain.

Continuous, hybrid dynamical systems, damped oscillator models and decentralised supply chain models are able to consider a time-varying inflow of orders and production. In comparison to this, queueing networks assume a given distribution for the inter-arrival times between consecutive orders and production times. Fluid models are based on the mean values of the distribution of these variables. Production processes are carried out at various locations of a supply chain. Continuous and hybrid dynamical systems as well as fluid models are based on a continuous production. In the case of fluid models the production rate for each product type is fixed. An adaptive production rate can be considered by continuous, hybrid dynamical systems and damped oscillator models (Helbing and Lämmer, 2005; Dashkovskiy et al., 2011). By contrast queueing networks and decentralised supply chain models are based on the flow of discrete products. In queueing networks one location produces different kinds of products, hence it needs to allocate its capacity to the production of the individual products (Scholz-Reiter et al., 2010; Dachkovski et al., 2005). Service disciplines like FIFO, priority and processor sharing are embedded in the modelling concept of queueing networks and fluid models. It is also possible to incorporate these in continuous and hybrid dynamical systems. Decentralised supply chain models allow the specification of various order policies. If a product needs to repeat a production step this can be modelled by all considered modelling approaches, since the approaches have the capability to model reentrant systems.

The transportation of products is basically modelled by the connections between the locations. Hybrid dynamical systems, decentralised supply chain models and queueing networks can model a discrete flow of intermediate and finished products. If for instance a truck with new intermediate products arrives, this leads to a jump of the work in progress of the considered location. In comparison to this continuous dynamical systems, damped oscillator models and fluid models capture the transportation by continuous material flows. In order to model transportation times continuous, hybrid dynamical systems and decentralised supply chain models can be used (Polushin et al., 2006).

The stability of a queueing network can be considered by analysing the corresponding fluid model. An advantage of fluid models is that their analysis can be interpreted directly in the context of the stochastic system. In the literature some methods exist to optimise the networks processes (Nazarathy and Weiss, 2009). Furthermore the stability radius of a fluid model quantifies the robustness of the considered supply chain. In this context the stability radius is the size of the smallest perturbation that destabilises the whole logistics network (Scholz-Reiter et al., 2011). However, only a few methods exist to design the network. Lefeber and Rooda (2008) designed for a given system with optimal steady state behaviour a controller such that the network converges towards the desired behaviour. In Scholz-Reiter et al. (2011) an approach for robust network design based on fluid models is presented, which utilises the stability radius. Damped oscillator models and decentralised supply chains allow to quantify the bullwhip effect. The stability of continuous and hybrid dynamical systems can be investigated with the concept of input-to-state stability. In particular the obtained gains provide an estimate for the robustness of the considered dynamical system.

4 Conclusions

In this paper we have discussed different routes to the modelling of supply chains. Damped oscillator models use the idea from physics and are derived from flow-balance equations. For such models linearisation techniques are used to determine local stability properties. In the discrete-time models of decentralised supply chains classical notions of L^2-gains (or RSME amplification factors) are used to describe stability properties. These approaches are extended to the more general nonlinear or hybrid nonlinear models in the input-to-state stability framework. In particular hybrid models are well suited for supply chains because different processes may be continuous or discrete in nature. A different approach is given by queueing networks, which consider explicitly the stochastic nature of processes which take place in supply chains. Stability criteria for such networks are given in terms of continuous and deterministic approximations. Our comparison shows that no dominant modelling approach exists and that the choice strongly depends on the characteristics of the supply chain. Further modelling frameworks use e.g., transport equations in partial differential equation form or tools closer to computer science. A full scale comparison of modelling tools still has to be undertaken.

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