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Peter Maass Pham Q. Muoi

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Peter Maass[†] and Pham. Q. Muoi[†]

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[†] Center for Industrial Mathematics, University of Bremen,
Bibliothekstr. 1, D-28334 Bremen, Germany
Email: pmaass@math.uni-bremen.de, pham@math.uni-bremen.de

Abstract

In this paper, we investigate sparsity regularization for the diffusion coefficient identification problem. Here, the regularization method is incorporated with the energy functional approach. The advantages of our approach are to deal with convex minimization problems. Therefore, the well-posedness of the problem is obtained without requiring regularity property of the parameter. The convexity of regularized problems also allows to use the fast algorithms developed recently. Furthermore, the convergence rates of the method are obtained under a simple source condition.

The main results of the paper are the well-posedness and convergence rates of sparsity regularization. We also obtain some new results of the continuity and the differentiability of related operators.

Keywords: *Sparsity regularization, Diffusion coefficient problem.*

1 Introduction

The diffusion coefficient identification problem is to identify the coefficient σ in the equation

$$-\operatorname{div}(\sigma\nabla\phi) = y \text{ in } \Omega,\ \phi = 0 \text{ on } \partial\Omega \tag{1}$$

from noisy data $\phi^\delta \in H_0^1(\Omega)$ of ϕ such that

$$\left\|\phi^* - \phi^\delta\right\|_{H^1(\Omega)} \le \delta. \quad (\delta > 0)$$

This problem has attracted great attention of many researchers. For surveys on this problem, we refer to [14, 34, 10, 22, 28, 23, 7, 32, 1, 6] and the references therein. It is well-known that the problem is ill-posed and thus need to be regularized. There have been several regularization methods proposed. Among of them, Tikhonov regularization [14, 9] and the total variational regularization [34, 5] are most popular.

In some applications, the coefficient σ^*, which needs to be recovered, has a sparse presentation, i.e. the number of nonzero components of $\sigma^* - \sigma^0$ are finite in an orthonormal basis (or frame) of $L^2(\Omega)$. The sparsity of $\sigma^* - \sigma^0$ promotes to use sparsity regularization.

Sparsity regularization has been of interest by many researchers for the last years. The well-posedness and some convergence rates of the method have been analyzed for linear inverse problems [8] as well as for nonlinear inverse problems [13]. Some numerical algorithms have also been proposed [24, 8, 4, 3, 27, 2]. It is shown that sparsity regularization is simple for use and very efficient for inverse problems with sparse solutions. This method has been investigated and applied very successfully to some fields such as for compressive imaging [11, 30, 31, 33]and electrical impedance tomography [20, 12, 19].

Note that it is possible to apply the least squares approach in [13] for our problem. However, it is not clear that the operator $F_D(\cdot)\,y$, the solution operator of (1), is weakly sequentially closed in $L^2(\Omega)$ without additional conditions. Therefore, if the least squares approach in [13] is applied, it needs further conditions. Moreover, this approach leads to a non-convex minimization problem and the source conditions are difficult to be checked for the problem, see e.g. [14].

To overcome this shortcoming, we use the energy functional approach incorporating with sparsity regularization, i.e. considering the minimization problem

$$\min_{\sigma\in\mathcal{A}} F_{\phi^\delta}(\sigma) + \alpha\Phi\left(\sigma - \sigma^0\right), \tag{2}$$

where $\mathcal{A}$ is an admissible set defined by

$$\mathcal{A} = \left\{\sigma \in L^\infty(\Omega) : \lambda \le \sigma \le \lambda^{-1} \text{ a.e. on } \Omega,\ \operatorname{supp}\left(\sigma - \sigma^0\right) \subset \Omega' \subset\subset \Omega\right\}, \tag{3}$$

with a given constant $\lambda \in (0,1)$ and Ω' being an open set with the smooth boundary that contained compactly in Ω, $\alpha > 0$ is a regularization parameter, σ^0 is the background value of σ, and

$$F_{\phi^\delta}(\sigma) := \int_\Omega \sigma \left|\nabla\left(F_D(\sigma) y - \phi^\delta\right)\right|^2 dx, \tag{4}$$

$$\Phi(\vartheta) := \sum \omega_k \left|\langle\vartheta, \varphi_k\rangle\right|^p, \quad (1 \le p \le 2) \tag{5}$$

where $\{\varphi_k\}$ is an orthonormal basis (or frame) of $L^2(\Omega)$ and $\omega_k \ge \omega_{min} > 0$ for all k.

We will prove that problem (2) is convex and well-posed, and under the condition that there exists w^* such that $\xi = \left(F_D'(\sigma^+) y\right)^* w^* \in \partial\Phi\left(\sigma^+ - \sigma^0\right)$, the convergence rates

$$D_\xi\left(\sigma^p_{\alpha,\delta}, \sigma^+\right) = O(\delta) \text{ and } \left\|\sigma^p_{\alpha,\delta} - \sigma^+\right\|_{L^2(\Omega)} = O\left(\sqrt{\delta}\right) \quad (1 < p \le 2),$$

are obtained as $\delta \to 0$ and $\alpha \sim \delta$. Here, $\sigma^p_{\alpha,\delta}$ is a minimizer of (2) and σ^+ is a Φ-minimizing solution of the diffusion coefficient identification problem.

Comparing the standard conditions in [13] and the references therein, our source condition is very simple and does not require the smallness. Furthermore, the objective functional in (2) is now convex and thus its global minimizers are easy to find and some efficient algorithms for convex functionals can be applied, see e.g. [24].

Note that the energy functional approach was first introduced by Zou [34] and then was used by Knowles in [21]. However, the authors in those papers did not consider the well-posedness and convergence rates of regularization methods. Recently, Hao and Quyen have used this approach incorporating with either Tikhonov regularization or the total variation regularization for some problems [14, 16, 15, 17]. In the following, we follows the outline of [14] and use the techniques in [14, 16] for obtaining the convergence rates of the method.

2 Auxiliary Results

We recall that a function ϕ in $H_0^1(\Omega)$ is a weak solution of (1) if the identity

$$\int_\Omega \sigma\nabla\phi\cdot\nabla v dx = \int_\Omega y v dx \tag{6}$$

holds for all $v \in H_0^1(\Omega)$.

If $\sigma \in \mathcal{A}$ and $y \in L^2(\Omega)$, then there is a unique weak solution $\phi \in H_0^1(\Omega)$ of (1) [14], which satisfies the inequality

$$\|\phi\|_{H^1(\Omega)} \le \frac{1}{C}\|y\|_{L^2(\Omega)}, \tag{7}$$

where $C > 0$ is a constant depending only on Ω and λ.

In the next sections, two following inequalities are used:

- For any $\eta \in H_0^1(\Omega)$ and $\sigma \in \mathcal{A}$, in virtue of the Poincaré-Friedrichs inequality we have

$$\int_\Omega \sigma|\nabla\eta|^2 dx \ge C\|\eta\|^2_{H^1(\Omega)} \tag{8}$$

with $C > 0$ defined by (7).

- For any $y \in L^r(\Omega), r \geq 2$ with a bounded set $\Omega \subset \mathbb{R}^d$, we have

$$\|y\|_{L^2(\Omega)} \leq |\Omega|^{\frac{1}{2}-\frac{1}{r}} \|y\|_{L^r(\Omega)}. \tag{9}$$

We shall endow the set $\mathcal{A}$ with the $L^q(\Omega)$ −norm, $q \in [1, \infty)$ and define the nonlinear coefficient-to-solution mapping $F_D(\cdot) y : \mathcal{A} \subset L^q(\Omega) \to H_0^1(\Omega)$ which maps the coefficient $\sigma \in \mathcal{A}$ to the solution $u = F_D(\sigma) y$ of problem (1).

Before considering sparsity regularization for the problem, we analyze some properties of $F_D(\cdot) y$ and $F_{\phi^\delta}(\cdot)$ with respect to the L^q−norm. These properties are needed for investigating the well-posedness and convergence rates of the method as well as numerical algorithms. They are derived by exploiting Meyers' gradient estimate [25], which has recently been employed by [29, 19].

Theorem 1 (Meyers' theorem) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ $(d \geq 2)$. Assume that $\sigma \in L^\infty(\Omega)$ satisfies $\lambda < \sigma < \lambda^{-1}$ for some fixed $\lambda \in (0, 1)$. For $z \in (L^r(\Omega))^d$ and $y \in L^r(\Omega)$, let $\phi \in H^1(\Omega)$ be a weak solution of the equation*

$$-\operatorname{div}(\sigma \nabla \phi) = -\operatorname{div}(z) + y \text{ in } \Omega.$$

Then, there exists a constant $Q \in (2, +\infty)$ depending on λ and d only, $Q \to 2$ as $\lambda \to 0$ and $Q \to \infty$ as $\lambda \to 1$, such that for any $2 < r < Q$, $\phi \in W^{1,r}_{loc}(\Omega)$ and for any $\Omega' \subset\subset \Omega$

$$\|\nabla \phi\|_{L^r(\Omega')} \leq C' \left(\|\phi\|_{H^1(\Omega)} + \|z\|_{L^r(\Omega)} + \|y\|_{L^r(\Omega)} \right),$$

where the constant C' depends on λ, d, r, Ω' and Ω.

Using this result, we can show that the mappings $F_D(\cdot) y$ and $F_{\phi^\delta}(\cdot)$ are continuous and continuous Fréchet differentiable on the set $\mathcal{A}$ with respect to the L^q-norm. These results are shown in the following lemmas.

Lemma 2 *Let $q \in \left(\frac{2Q}{Q-2}, \infty\right]$, $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$ and $y \in L^r(\Omega)$. For $\sigma, \sigma + \vartheta \in \mathcal{A}$, we have*

$$\|\nabla F_D(\sigma + \vartheta) y - \nabla F_D(\sigma) y\|_{L^2(\Omega)} \leq C \|\vartheta\|_{L^q(\Omega')} \|y\|_{L^r(\Omega)},$$

where C is a positive constant.

Proof. The weak solution formulas of $F_D(\sigma) y$ and $F_D(\sigma + \vartheta) y$ give

$$\int_\Omega \sigma \nabla F_D(\sigma) y \cdot \nabla v dx = \int_\Omega (\sigma + \vartheta) \nabla F_D(\sigma + \vartheta) y \cdot \nabla v dx, \ \forall v \in H_0^1(\Omega),$$

i.e.

$$\int_\Omega \sigma \nabla (F_D(\sigma + \vartheta) y - F_D(\sigma) y) \cdot \nabla v dx = -\int_\Omega \vartheta \nabla F_D(\sigma + \vartheta) y \cdot \nabla v dx, \ \forall v \in H_0^1(\Omega).$$

Taking $v = F_D(\sigma + \vartheta) y - F_D(\sigma) y \in H_0^1(\Omega)$ in the last equation, we obtain

$$\begin{aligned}\int_\Omega \sigma |\nabla (F_D(\sigma + \vartheta) y - F_D(\sigma) y)|^2 dx &= -\int_\Omega \vartheta \nabla F_D(\sigma + \vartheta) y \cdot \nabla (F_D(\sigma + \vartheta) y - F_D(\sigma) y) dx \\ &= -\int_{\Omega'} \vartheta \nabla F_D(\sigma + \vartheta) y \cdot \nabla (F_D(\sigma + \vartheta) y - F_D(\sigma) y) dx \\ &\leq \|\vartheta\|_{L^q(\Omega')} \|\nabla F_D(\sigma + \vartheta) y\|_{L^r(\Omega')} \|\nabla (F_D(\sigma + \vartheta) y - F_D(\sigma) y)\|_{L^2(\Omega)},\end{aligned}$$

where $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$. The assumption $q \in \left(\frac{2Q}{Q-2}, \infty\right]$ implies that $r \in (2, Q)$. By Theorem 1, there exist constants C and C' such that

$$\|\nabla F_D(\sigma + \vartheta) y\|_{L^r(\Omega')} \leq C' \left(\|F_D(\sigma + \vartheta) y\|_{H^1(\Omega)} + \|y\|_{L^r(\Omega)} \right) \overset{(7),(9)}{\leq} C \|y\|_{L^r(\Omega)}.$$

It follows that there exists a constant C such that

$$\|\nabla F_D(\sigma + \vartheta) y - \nabla F_D(\sigma) y\|_{L^2(\Omega)} \leq C \|\vartheta\|_{L^q(\Omega')} \|y\|_{L^r(\Omega)}.$$

∎

Remark 3 *1) Note that for $\sigma, \sigma+\vartheta \in \mathcal{A}$ and $1 \le q_1 \le q_2$, we have*

$$|\Omega|^{-1/q_1} \|\vartheta\|_{L^{q_1}(\Omega)} \le |\Omega|^{-1/q_2} \|\vartheta\|_{L^{q_2}(\Omega)},$$

and

$$\|\vartheta\|^{q_2}_{L^{q_2}(\Omega)} \le \left(2\lambda^{-1}\right)^{q_2-q_1} \|\vartheta\|^{q_1}_{L^{q_1}(\Omega)}.$$

This means that the convergence of ϑ to zero with respect to the $L^{q_1}(\Omega)-$norm and the $L^{q_2}(\Omega)-$norm are equivalent.

2) By the above lemma, $F_D(\cdot)\,y$ is Lipschitz continuous on $\mathcal{A}$ with respect to the $L^q(\Omega)-$norm for $q \in \left(\frac{2Q}{Q-2}, \infty\right]$. Furthermore, by the above remark, it implies that $F_D(\cdot)\,y$ is continuous on $\mathcal{A}$ with respect to the $L^q(\Omega)$-norm for any $q \ge 1$.

Lemma 4 *Let $q \in \left(\frac{2Q}{Q-2}, \infty\right]$, $\frac{1}{q}+\frac{1}{r}=\frac{1}{2}$ and $y \in L^{r+\epsilon}(\Omega)$ with some $\epsilon > 0$. Then, the mapping $F_D(\cdot)\,y : \mathcal{A} \subset L^q(\Omega) \to H^1_0(\Omega)$ is continuously Fréchet differentiable on $\mathcal{A}$ and for each $\sigma \in \mathcal{A}$, the Fréchet derivative $F'_D(\sigma)\,y$ of $F_D(\cdot)\,y$ has the property that the differential $\eta := F'_D(\sigma)\,y(\vartheta)$, with any $\vartheta \in L^\infty(\Omega')$ extended by zero outside Ω', is the (unique) weak solution of the Dirichlet problem*

$$-\operatorname{div}(\sigma\nabla\eta) = \operatorname{div}(\vartheta\nabla F_D(\sigma)\,y) \ \text{ in } \Omega,\ \eta = 0 \text{ on } \partial\Omega$$

in the sense that it satisfies the equation

$$\int_\Omega \sigma\nabla F'_D(\sigma)\,y(\vartheta)\cdot\nabla v dx = -\int_\Omega \vartheta\nabla F_D(\sigma)\,y\cdot\nabla v dx \tag{10}$$

for all $v \in H^1_0(\Omega)$. Moreover,

$$\|F'_D(\sigma)\,y(\vartheta)\|_{H^1(\Omega)} \le C_1 \|y\|_{L^r(\Omega)} \|\vartheta\|_{L^q(\Omega')}, \forall\vartheta \in L^\infty(\Omega'), \tag{11}$$

where C_1 is a positive constant.

Proof. Note that variational equation (10) has the unique solution $\eta := \eta(\vartheta) = F'_D(\sigma)\,y(\vartheta) \in H^1_0(\Omega)$ with $\sigma \in \mathcal{A}$. We first show that for a fixed σ in $\mathcal{A}$, $\eta = \eta(\vartheta)$ defines a bounded linear operator from $L^q(\Omega')$ to $H^1_0(\Omega)$ for any $q \in \left(\frac{2Q}{Q-2}, \infty\right]$. From (10), η is a linear operator of ϑ. By the weak solution formula of η and the generalized Hölder inequality, we have

$$\begin{aligned}\int_\Omega \sigma\nabla\eta\cdot\nabla\eta dx &= -\int_\Omega \vartheta\nabla F_D(\sigma)\,y\cdot\nabla\eta dx\\ &= -\int_{\Omega'} \vartheta\nabla F_D(\sigma)\,y\cdot\nabla\eta dx\\ &\le \|\vartheta\|_{L^q(\Omega')} \|\nabla F_D(\sigma)\,y\|_{L^r(\Omega')} \|\nabla\eta\|_{L^2(\Omega)}.\end{aligned}$$

From the last inequality and (8), there exists a constant C such that

$$\|\eta\|_{H^1(\Omega)} \le C\|\vartheta\|_{L^q(\Omega')} \|\nabla F_D(\sigma)\,y\|_{L^r(\Omega')}. \tag{12}$$

Besides, the assumption $q \in \left(\frac{2Q}{Q-2}, \infty\right]$ implies $r \in (2, Q)$. By Theorem 1, (7) and (9), there exist positive constants C, C', C'' such that

$$\begin{aligned}\|\nabla F_D(\sigma)\,y\|_{L^r(\Omega')} &\le C'\left(\|F_D(\sigma)\,y\|_{H^1(\Omega)} + \|y\|_{L^r(\Omega)}\right)\\ &\le C'\left(\frac{1}{C}\|y\|_{L^2(\Omega)} + \|y\|_{L^r(\Omega)}\right)\\ &\le C''\|y\|_{L^r(\Omega)}.\end{aligned} \tag{13}$$

Thus, due to two last inequalities, η is a bounded linear operator from $L^q(\Omega') \to H^1_0(\Omega)$ and there exists a positive constant C_1 such that

$$\|F'_D(\sigma)\,y(\vartheta)\|_{H^1(\Omega)} \le C_1\|y\|_{L^r(\Omega)} \|\vartheta\|_{L^q(\Omega')}, \forall\vartheta \in L^\infty(\Omega').$$

We now show that $F_D(\cdot)\,y$ is Fréchet differentiable. Note that the function $R := F_D(\sigma+\vartheta)\,y - F_D(\sigma)\,y - \eta \in H_0^1(\Omega)$ is the weak solution of the equation

$$-\operatorname{div}\left(\left(\sigma+\vartheta\right)\nabla R\right) = \operatorname{div}\left(\vartheta\nabla\eta\right) \text{ in } \Omega.$$

Taking R as the test function in the weak solution formula of R gives

$$\begin{aligned}\int_\Omega (\sigma+\vartheta)\left|\nabla R\right|^2 dx &= -\int_\Omega \vartheta\nabla\eta\cdot\nabla R dx = -\int_{\Omega'} \vartheta\nabla\eta\cdot\nabla R dx \\ &\leq \left\|\vartheta\right\|_{L^q(\Omega')}\left\|\nabla\eta\right\|_{L^r(\Omega')}\left\|\nabla R\right\|_{L^2(\Omega)}.\end{aligned}$$

This implies that

$$\frac{\left\|R\right\|_{H^1(\Omega)}}{\left\|\vartheta\right\|_{L^q(\Omega')}} \leq C\left\|\nabla\eta\right\|_{L^r(\Omega')}. \tag{14}$$

To show that $F_D(\cdot)\,y : \mathcal{A}\subset L^q(\Omega)\to H_0^1(\Omega)$ is continuously Fréchet differentiable and its differential $F_D'(\sigma)\,y\,(\vartheta)$ is η, we need to prove that $\left\|\nabla\eta\right\|_{L^r(\Omega')}$ converges to zero as $\left\|\vartheta\right\|_{L^q(\Omega')}$ converges to zero.

By Theorem 1, there exists a positive constant C such that

$$\left\|\nabla\eta\right\|_{L^r(\Omega')} \leq C\left(\left\|\eta\right\|_{H^1(\Omega)} + \left\|\vartheta\nabla F_D(\sigma)\,y\right\|_{L^r(\Omega')}\right).$$

Since $\left\|\eta\right\|_{H^1(\Omega)}$ converges to zero as $\left\|\vartheta\right\|_{L^q(\Omega')}$ converges to zero by (12), we need to prove that

$$\left\|\vartheta\nabla F_D(\sigma)\,y\right\|_{L^r(\Omega')} \to 0.$$

Take any small $\epsilon_1\in(0,\epsilon)$ such that $r' = r+\epsilon_1\in(r,Q)$. Using Hölder's inequality, we deduce

$$\begin{aligned}\int_{\Omega'}\left|\vartheta\nabla F_D(\sigma)\,y\right|^r dx &= \int_{\Omega'}\left|\vartheta\right|^r\left|\nabla F_D(\sigma)\,y\right|^r dx \\ &\leq \left(\int_{\Omega'}\left|\vartheta\right|^{\frac{rr'}{r'-r}}dx\right)^{1-\frac{r}{r'}}\left(\int_{\Omega'}\left|\nabla F_D(\sigma)\,y\right|^{r'}dx\right)^{\frac{r}{r'}}. \\ &\leq C_2\left\|y\right\|_{L^{r'}(\Omega)}^r\left(\int_{\Omega'}\left|\vartheta\right|^{\frac{rr'}{r'-r}}dx\right)^{1-\frac{r}{r'}},\end{aligned} \tag{15}$$

where we have applied Theorem 1 to the term $\left\|\nabla F_D(\sigma)\,y\right\|_{L^{r'}(\Omega')}$, see (13). By Remark 3, the convergence of ϑ to zero with respect to the $L^{q_1}(\Omega)-$norm and the $L^{q_2}(\Omega)-$norm ($q_1,q_2\in[1,\infty)$) are equivalent. Therefore, $\left\|\vartheta\nabla F_D(\sigma)\,y\right\|_{L^r(\Omega')}$ converges to zero as $\left\|\vartheta\right\|_{L^q(\Omega')}$ converges to zero. ■

Remark 5 *1) If $y\in L^r(\Omega)$, then from the proof above we conclude that $F_D(\cdot)\,y : \mathcal{A}\subset L^q(\Omega)\to H_0^1(\Omega)$ is Gâuteaux differentiable.*

2) This lemma improves the known results on the differentiability of $F_D(\cdot)\,y$ with respect to the $L^\infty-$norm in [21, 14]. There, the authors have shown that $F_D(\cdot)\,y : \mathcal{A}\subset L^\infty(\Omega)\to H_0^1(\Omega)$ is the Fréchet differentiable under the condition $y\in L^\infty(\Omega)$ [21] or $y\in L^2(\Omega)$ [14].

Lemma 6 *For $\phi\in H_0^1(\Omega)$, the functional $F_\phi(\cdot) : \mathcal{A}\subset L^q(\Omega)\to\mathbb{R}$ defined by*

$$F_\phi(\sigma) = \int_\Omega \sigma\left|\nabla\left(F_D(\sigma)\,y-\phi\right)\right|^2 dx$$

has the following properties

1) For $q\geq 1$ and $y\in L^r(\Omega)$, $F_\phi(\cdot)$ is continuous with respect to the L^q-norm.

2) For $q\in\left(\frac{2Q}{Q-2},\infty\right]$, $\frac{1}{q}+\frac{1}{r}=\frac{1}{2}$ and $y\in L^{r+\epsilon}(\Omega)$ with $\epsilon>0$, $F_\phi(\cdot)$ is Fréchet differentiable with respect to the L^q-norm and

$$F_\phi'(\sigma)\,\vartheta = -\int_\Omega \vartheta\left(\left|\nabla F_D(\sigma)\,y\right|^2 - \left|\nabla\phi\right|^2\right)dx.$$

Furthermore, $F_\phi(\cdot)$ is convex on the convex set $\mathcal{A}$ and $F_\phi''(\cdot)$ is uniformly bounded.

Proof. 1) We first prove for $q \in \left(\frac{2Q}{Q-2}, \infty\right]$. For $\sigma, \sigma+\vartheta \in \mathcal{A}$, we have

$$
\begin{aligned}
&F_\phi(\sigma+\vartheta) - F_\phi(\sigma)\\
&= \int_\Omega (\sigma+\vartheta)\left|\nabla\left(F_D(\sigma+\vartheta)y-\phi\right)\right|^2 - \sigma\left|\nabla\left(F_D(\sigma)y-\phi\right)\right|^2 dx\\
&= \int_\Omega \sigma\left(\left|\nabla\left(F_D(\sigma+\vartheta)y-\phi\right)\right|^2 - \left|\nabla\left(F_D(\sigma)y-\phi\right)\right|^2\right) dx + \int_\Omega \vartheta\left|\nabla\left(F_D(\sigma+\vartheta)y-\phi\right)\right|^2 dx.
\end{aligned}
$$

Using the triangle inequality, generalized Hölder inequality and Theorem 1, the second term is estimated by

$$
\begin{aligned}
&\int_\Omega \vartheta\left|\nabla\left(F_D(\sigma+\vartheta)y-\phi\right)\right|^2 dx = \int_{\Omega'} \vartheta\left|\nabla\left(F_D(\sigma+\vartheta)y-\phi\right)\right|^2 dx\\
&\le \|\vartheta\|_{L^q(\Omega')}\left\|\nabla\left(F_D(\sigma+\vartheta)y-\phi\right)\right\|_{L^2(\Omega)}\left(\|\nabla F_D(\sigma+\vartheta)y\|_{L^r(\Omega')} + \|\nabla\phi\|_{L^r(\Omega')}\right)\\
&\le C\|\vartheta\|_{L^q(\Omega')}.
\end{aligned}
$$

On the other hand, by Lemma 2 the first term is estimated by

$$
\begin{aligned}
&\int_\Omega \sigma\left(\left|\nabla\left(F_D(\sigma+\vartheta)y-\phi\right)\right|^2 - \left|\nabla\left(F_D(\sigma)y-\phi\right)\right|^2\right) dx\\
&\le \lambda^{-1}\int_\Omega \nabla\left(F_D(\sigma+\vartheta)y - F_D(\sigma)y\right)\cdot\nabla\left(F_D(\sigma+\vartheta)y + F_D(\sigma)y - 2\phi\right) dx\\
&\le C\left\|\nabla\left(F_D(\sigma+\vartheta)y - F_D(\sigma)y\right)\right\|_{L^2(\Omega)} \le C'\|\vartheta\|_{L^q(\Omega')}.
\end{aligned}
$$

Therefore, $F_\phi(\cdot)$ is Lipschitz continuous on $\mathcal{A}$ with respect to the $L^q(\Omega')$-norm for $q \in \left(\frac{2Q}{Q-2}, \infty\right]$.

Finally, by Remark 3 F_ϕ is continuous on $\mathcal{A}$ with respect to the $L^q(\Omega')$ $-$norm for $q \ge 1$.

2) From Lemma 4, it implies that $F_\phi(\cdot)$ is Fréchet differentiable and

$$
F'_\phi(\sigma)\vartheta = \int_\Omega \vartheta\left|\nabla\left(F_D(\sigma)y-\phi\right)\right|^2 dx + 2\int_\Omega \sigma\nabla\left(F_D(\sigma)y-\phi\right)\cdot\nabla F'_D(\sigma)\vartheta dx.
$$

Since $F_D(\sigma)y - \phi \in H_0^1(\Omega)$ and (10), the last equation yields

$$
\begin{aligned}
F'_\phi(\sigma)\vartheta &= \int_\Omega \vartheta\left|\nabla\left(F_D(\sigma)y-\phi\right)\right|^2 dx - 2\int_\Omega \vartheta\nabla F_D(\sigma)y\cdot\nabla\left(F_D(\sigma)y-\phi\right) dx\\
&= -\int_\Omega \vartheta\left(\left|\nabla F_D(\sigma)y\right|^2 - |\nabla\phi|^2\right) dx.
\end{aligned}
$$

For $\vartheta \in L^\infty(\Omega')$ and extended by zero outside Ω', the second derivative of $F_\phi(\cdot)$ is given by

$$
F''_\phi(\sigma)(\vartheta,\vartheta) = -2\int_\Omega \vartheta\nabla F_D(\sigma)y\cdot\nabla F'_D(\sigma)y(\vartheta)dx = 2\int_\Omega \sigma\left|\nabla F'_D(\sigma)y(\vartheta)\right|^2 dx \ge 0.
$$

Therefore, $F_\phi(\cdot)$ is convex. Furthermore, by Lemma 4, it implies that $F''_\phi(\cdot)$ is uniformly bounded on $\mathcal{A}$. ■

Remark 7 *The uniform boundedness of $F''_\phi(\cdot)$ implies that $F'_\phi(\cdot)$ is Lipschitz continuous with respect to the L^q-norms with $q \in \left(\frac{2Q}{Q-2}, \infty\right]$.*

3 The Well-posedness

We now assume that there exists some $\sigma^* \in \mathcal{A}$ such that $\phi^* = F_D(\sigma^*)y$ and only noisy data $\phi^\delta \in H_0^1(\Omega)$ of ϕ^* such that

$$
\left\|\phi^* - \phi^\delta\right\|_{H^1(\Omega)} \le \delta
$$

with $\delta > 0$ are given. Our problem is to reconstruct σ^* from ϕ^δ. Because of the ill-posedness of the problem and the assumption of sparsity of $\sigma^* - \sigma^0$, using sparsity regularization incorporated with the energy functional approach leads to considering the minimization problem (2).

We now analyze the well-posedness of problem (2), which consists of the existence, stability and convergence. Before proving the main results, we introduce some properties of the functional (5) and the notion of Φ-minimizing solution.

Lemma 8 *The functional Φ defined by (5) has the following properties*

1) *Φ is non-negative, convex and weakly lower semi-continuous.*

2) *There exists a positive constant C such that for any $u \in \mathcal{H}$,*

$$\Phi(u) \geq \omega_{min} C^{p/2} \|u\|^p .$$

This implies that Φ is weakly coercive, i.e. $\Phi(u) \to \infty$ as $\|u\| \to \infty$.

3) *If $\{u^n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ weakly converges to $u \in \mathcal{H}$ and $\Phi(u^n)$ converges to $\Phi(u)$, then $\Phi(u^n - u)$ converges to zero.*

Proof. Φ is non-negative, convex and weakly lower semi-continuous because it is the sum of non-negative, convex and weakly continuous functionals. The proofs of 2) and 3) can be found in [13, Remark 3.] and [13, Lemma 2.], respectively. ■

Lemma 9 *The set*

$$\Pi(\phi^*) := \{\sigma \in \mathcal{A} : F_D(\sigma)\, y = \phi^*\}$$

is nonempty, convex, bounded and closed with respect to the $L^2(\Omega)$-norm. Thus, there exists a solution σ^+ of the problem

$$\min_{\sigma\in\Pi(\phi^*)} \Phi\left(\sigma - \sigma^0\right)$$

which is called a Φ-minimizing solution of the diffusion coefficient problem. The Φ-minimizing solution is unique if $p > 1$.

Proof. It is trivial that the set $\Pi(\phi^*)$ is nonempty, convex and bounded. The closeness of $\Pi(\phi^*)$ in the $L^2(\Omega)$ −norm is proven similarly as that of [14, Lemma 2.1].

We now prove that there exists at least a Φ-minimizing solution. Suppose that there does not exist a Φ-minimizing solution in $\Pi(\phi^*)$. There exists a sequence $\{\sigma^k\} \subset \Pi(\phi^*)$ such that $\Phi\left(\sigma^k - \sigma^0\right) \to c$ and

$$c < \Phi\left(\sigma - \sigma^0\right) \text{ for all } \sigma \in \Pi(\phi^*). \tag{16}$$

Since $\Pi(\phi^*)$ is weakly compact, there exists a subsequence of $\{\sigma^k\}$, denoted by $\{\sigma^k\}$ again, which weakly converges to $\tilde{\sigma} \in \Pi(\phi^*)$. From the weakly lower semi-continuity of Φ, it follows that

$$\Phi\left(\tilde{\sigma} - \sigma^0\right) \leq \lim_{k\to\infty} \inf \Phi\left(\sigma^k - \sigma^0\right) = c.$$

This gives a contradiction to (16).

For $p > 1$, $\Phi(\cdot)$ is strictly convex and thus the Φ-minimizing solution is unique. ■

Theorem 10 (Existence) *Problem (2) has at least one solution.*

Proof. Since the functional $F_{\phi^\delta}(\cdot)$ is convex and continuous with respect to the $L^2(\Omega)$-norm, it is weakly lower semi-continuous. Besides, $\Phi(\cdot)$ is also convex and weakly lower semi-continuous with respect to the $L^2(\Omega)$-norm (see Lemma 8). Therefore, the objective functional of problem (2) is convex and weakly lower semi-continuous on $\mathcal{A}$. On the other hand, since $\mathcal{A}$ is nonempty, convex, bounded and closed with respect to the $L^2(\Omega)$-norm, it is weakly compact. Therefore, there exists at least one solution of (2). ■

Theorem 11 (Stability) *For a fixed regularization $\alpha > 0$, let the sequence $\{\phi^n\}$ converge to ϕ^δ in $H_0^1(\Omega)$ and*

$$\sigma^n \in \operatorname*{argmin}_{\sigma\in\mathcal{A}} F_{\phi^n}(\sigma) + \alpha\Phi\left(\sigma - \sigma^0\right).$$

Then, there exist a subsequence $\{\sigma^{n_k}\}$ of $\{\sigma^n\}$ and a minimizer $\sigma^p_{\alpha,\delta}$ of (2) such that

$$\left\|\sigma^{n_k} - \sigma^p_{\alpha,\delta}\right\|_{L^2(\Omega)} \to 0.$$

In addition, if the minimizer $\sigma^p_{\alpha,\delta}$ is unique, then the sequence $\{\sigma^n\}$ converges to $\sigma^p_{\alpha,\delta}$ with respect to the $L^2(\Omega)$-norm.

Proof. By the definition of σ^n, we have

$$\begin{aligned} F_{\phi^n}\left(\sigma^n\right)+\alpha\Phi\left(\sigma^n-\sigma^0\right) &\le F_{\phi^n}\left(\sigma\right)+\alpha\Phi\left(\sigma-\sigma^0\right) \\ &\le \lambda^{-1}\left(\left\|F_D\left(\sigma\right)y\right\|^2_{H^1(\Omega)}+C\right)+\alpha\Phi\left(\sigma-\sigma^0\right) \end{aligned} \tag{17}$$

for any $\sigma\in\mathcal{A}$, where the constant C is independent of n such that $\|\phi^n\|^2_{H^1(\Omega)}\le C$ for all n. This follows that $\left\{\Phi\left(\sigma^n-\sigma^0\right)\right\}$ is bounded. Since Φ is weakly coercive in $L^2\left(\Omega\right)$ (see Lemma 8), the sequence $\{\sigma^n\}$ is also bounded in $L^2\left(\Omega\right)$. Therefore, there exist a subsequence of $\{\sigma^n\}$ denoted by $\{\sigma^{n_k}\}$ and an element $\sigma^p_{\alpha,\delta}\in L^2\left(\Omega\right)$ such that $\{\sigma^{n_k}\}$ weakly converges to $\sigma^p_{\alpha,\delta}$ in $L^2\left(\Omega\right)$. Since $\mathcal{A}$ is a convex closed set in $L^2\left(\Omega\right)$, $\sigma^p_{\alpha,\delta}\in\mathcal{A}$. On the other hand, since $F_{\phi^\delta}\left(\cdot\right)$ and $\Phi\left(\cdot\right)$ are weakly lower semi-continuous, we have

$$F_{\phi^\delta}\left(\sigma^p_{\alpha,\delta}\right)\le\liminf_k F_{\phi^\delta}\left(\sigma^{n_k}\right) \tag{18}$$

and

$$\Phi\left(\sigma^p_{\alpha,\delta}-\sigma^0\right)\le\liminf_k \Phi\left(\sigma^{n_k}-\sigma^0\right). \tag{19}$$

Furthermore, we have

$$\begin{aligned} F_{\phi^\delta}\left(\sigma^{n_k}\right)=F_{\phi^{n_k}}\left(\sigma^{n_k}\right)+\Bigg(2\int_\Omega\sigma^{n_k}\nabla F_D\left(\sigma^{n_k}\right)y\cdot\nabla\left(\phi^{n_k}-\phi^\delta\right)dx \\ -\int_\Omega\sigma^{n_k}\left|\nabla\left(\phi^{n_k}-\phi^\delta\right)\right|^2dx\Bigg). \end{aligned} \tag{20}$$

Since $\phi^{n_k}\to\phi^\delta$ in $H^1\left(\Omega\right)$, the term in brackets on the right-hand side of (20) converges to zero as $k\to\infty$. Therefore,

$$\liminf_k F_{\phi^\delta}\left(\sigma^{n_k}\right)=\liminf_k F_{\phi^{n_k}}\left(\sigma^{n_k}\right),\ \limsup_k F_{\phi^\delta}\left(\sigma^{n_k}\right)=\limsup_k F_{\phi^{n_k}}\left(\sigma^{n_k}\right). \tag{21}$$

From (21), (17), (18) and (19), we obtain

$$\begin{aligned} F_{\phi^\delta}\left(\sigma^p_{\alpha,\delta}\right)+\alpha\Phi\left(\sigma^p_{\alpha,\delta}-\sigma^0\right) &\overset{(18),(19)}{\le} \liminf_k F_{\phi^\delta}\left(\sigma^{n_k}\right)+\alpha\liminf_k\Phi\left(\sigma^{n_k}-\sigma^0\right) \\ &\overset{(21)}{\le} \liminf_k\left(F_{\phi^{n_k}}\left(\sigma^{n_k}\right)+\alpha\Phi\left(\sigma^{n_k}-\sigma^0\right)\right) \\ &\le \limsup_k\left(F_{\phi^{n_k}}\left(\sigma^{n_k}\right)+\alpha\Phi\left(\sigma^{n_k}-\sigma^0\right)\right) \\ &\overset{(17)}{\le} \limsup_k\left(F_{\phi^{n_k}}\left(\sigma\right)+\alpha\Phi\left(\sigma-\sigma^0\right)\right) \\ &= F_{\phi^\delta}\left(\sigma\right)+\alpha\Phi\left(\sigma-\sigma^0\right) \end{aligned} \tag{22}$$

for all $\sigma\in\mathcal{A}$. It means that $\sigma^p_{\alpha,\delta}$ is a minimizer of (2).

From (22), setting $\sigma=\sigma^p_{\alpha,\delta}$, we get

$$\lim_k\left(F_{\phi^\delta}\left(\sigma^{n_k}\right)+\alpha\Phi\left(\sigma^{n_k}-\sigma^0\right)\right)=F_{\phi^\delta}\left(\sigma^p_{\alpha,\delta}\right)+\alpha\Phi\left(\sigma^p_{\alpha,\delta}-\sigma^0\right).$$

Together with (18) and (19), we deduce that $\Phi\left(\sigma^{n_k}-\sigma^0\right)\to\Phi\left(\sigma^p_{\alpha,\delta}-\sigma^0\right)$. Finally, since $\{\sigma^{n_k}\}$ weakly converges to $\sigma^p_{\alpha,\delta}$ and $\Phi\left(\sigma^{n_k}-\sigma^0\right)\to\Phi\left(\sigma^p_{\alpha,\delta}-\sigma^0\right)$ as $k\to\infty$, we conclude that $\Phi\left(\sigma^{n_k}-\sigma^p_{\alpha,\delta}\right)\to 0$ as $k\to 0$, and thus $\left\|\sigma^{n_k}-\sigma^p_{\alpha,\delta}\right\|_{L^2(\Omega)}\to 0$ as $k\to\infty$ by Lemma 8.

In the case the minimizer $\sigma^p_{\alpha,\delta}$ is unique, the convergence of the original sequence $\{\sigma^n\}$ to $\sigma^p_{\alpha,\delta}$ follows by a subsequence argument. ■

Theorem 12 (Convergence) *Assume that the operator equation $F_D\left(\sigma\right)y=\phi^*$ attains a solution in $\mathcal{A}$ and that $\alpha:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ satisfies*

$$\alpha\left(\delta\right)\to 0 \text{ and } \frac{\delta^2}{\alpha\left(\delta\right)}\to 0 \text{ as } \delta\to 0.$$

Let $\delta_n \to 0$ and $\|\phi^n - \phi^*\|_{H^1(\Omega)} \le \delta_n$. Moreover, let $\alpha_n = \alpha(\delta_n)$ and

$$\sigma^n \in \operatorname*{argmin}_{\sigma\in\mathcal{A}} F_{\phi^n}(\sigma) + \alpha_n \Phi\left(\sigma - \sigma^0\right).$$

Then, there exist a Φ-minimizing solution σ^+ of $F_D(\sigma) y = \phi^*$ and a subsequence of $\{\sigma^n\}$ converging to σ^+ on $\mathcal{A}$ with respect to the $L^2(\Omega)-$norm.

Proof. Let $\tilde{\sigma} \in \mathcal{A}$ be a solution of $F_D(\sigma) y = \phi^*$. The definition of σ^n implies that

$$\begin{aligned}
F_{\phi^n}(\sigma^n) + \alpha_n \Phi\left(\sigma^n - \sigma^0\right) &\le F_{\phi^n}(\tilde{\sigma}) + \alpha_n \Phi\left(\tilde{\sigma} - \sigma^0\right) \\
&\le \frac{1}{\lambda}\int_\Omega \left|\nabla\left(F_D(\tilde{\sigma}) y - \phi^n\right)\right|^2 + \alpha_n \Phi\left(\tilde{\sigma} - \sigma^0\right) \\
&\le \frac{1}{\lambda}\|\phi^* - \phi^n\|^2_{H^1(\Omega)} + \alpha_n \Phi\left(\tilde{\sigma} - \sigma^0\right) \\
&\le \frac{1}{\lambda}\delta_n^2 + \alpha_n \Phi\left(\tilde{\sigma} - \sigma^0\right).
\end{aligned} \tag{23}$$

In particular, when $\delta \to 0$ and $\alpha \sim \delta^2$, it follows that

$$F_{\phi^n}(\sigma^n) \to 0 \text{ and } \limsup_n \Phi\left(\sigma^n - \sigma^0\right) \le \Phi\left(\tilde{\sigma} - \sigma^0\right). \tag{24}$$

This implies that $\{\Phi\left(\sigma^n - \sigma^0\right)\}$ is bounded. Since $\Phi(\cdot)$ is weakly coercive, $\{\sigma^n\}$ is bounded, too. Therefore, there exist a subsequence $\{\sigma^{n_k}\}$ of $\{\sigma^n\}$ and $\sigma^+ \in \mathcal{A}$ such that σ^{n_k} weakly converges to σ^+. From (24), we deduce

$$\begin{aligned}
F_{\phi^*}(\sigma^{n_k}) &= \int_\Omega \sigma^{n_k}\left|\nabla\left(F_D(\sigma^{n_k}) y - \phi^*\right)\right|^2 \\
&\le \int_\Omega \sigma^{n_k}\left|\nabla\left(F_D(\sigma^{n_k}) y - \phi^{n_k}\right)\right|^2 + \int_\Omega \sigma^{n_k}\left|\nabla\left(\phi^{n_k} - \phi^*\right)\right|^2 \\
&\le F_{\phi^{n_k}}(\sigma^{n_k}) + \lambda^{-1}\|\phi^{n_k} - \phi^*\|^2_{H^1(\Omega)} \to 0 \ (k \to \infty).
\end{aligned}$$

Since $F_{\phi^*}(\cdot)$ is weakly lower semi-continuous,

$$0 \le F_{\phi^*}\left(\sigma^+\right) \le \liminf_k F_{\phi^*}(\sigma^{n_k}) = 0.$$

Thus, $F_{\phi^*}(\sigma^+) = 0$. It implies that $\|F_D(\sigma^+) y - \phi^*\|_{H^1(\Omega)} = 0$. Hence σ^+ is a solution of the equation $F_D(\sigma) y = \phi^*$.

Moreover, since $\Phi(\cdot)$ is weakly lower semi-continuous in $L^2(\Omega)$, by using (24) we get

$$\Phi\left(\sigma^+ - \sigma^0\right) \le \liminf_k \Phi\left(\sigma^{n_k} - \sigma^0\right) \le \limsup_k \Phi\left(\sigma^{n_k} - \sigma^0\right) \le \Phi\left(\tilde{\sigma} - \sigma^0\right). \tag{25}$$

It implies that σ^+ is a Φ-minimizing solution. Finally, choosing $\tilde{\sigma} = \sigma^+$ in (25), we have $\Phi\left(\sigma^{n_k} - \sigma^0\right) \to \Phi\left(\sigma^+ - \sigma^0\right)$ as $k \to \infty$. Since $\{\sigma^{n_k} - \sigma^0\}$ weakly converges to $\sigma^+ - \sigma^0$ in $L^2(\Omega)$ and $\Phi\left(\sigma^{n_k} - \sigma^0\right) \to \Phi\left(\sigma^+ - \sigma^0\right)$ as $k \to \infty$, $\Phi(\sigma^{n_k} - \sigma^+) \to 0$ as $k \to 0$ and thus $\|\sigma^{n_k} - \sigma^+\|_{L^2(\Omega)} \to 0$.

In the case the minimizer σ^+ is unique, the convergence of the original sequence $\{\sigma^n - \sigma^0\}$ to $\sigma^+ - \sigma^0$ follows by a subsequence argument. ∎

4 Convergence Rates

As shown before, for $\sigma \in \mathcal{A}$, the operator

$$F_D'(\sigma) y(\cdot) : L^q(\Omega') \to H_0^1(\Omega) \text{ with } q \in \left(\frac{2Q}{Q-2}, \infty\right]$$

is continuous and linear. Denote by

$$\left(F_D'(\sigma) y\right)^*(\cdot) : H^{-1}(\Omega) = \left(H_0^1(\Omega)\right)^* \to L^{q_1}(\Omega') \text{ with } \frac{1}{q} + \frac{1}{q_1} = 1,$$

the dual operator of $F'_D(\sigma)\, y$. Then,

$$\left\langle \left(F'_D(\sigma)\, y\right)^* (w^*), \vartheta \right\rangle_{(L^{q_1}(\Omega'), L^q(\Omega'))} = \langle w^*, F'_D(\sigma)\, y(\vartheta)\rangle_{\left(H^{-1}(\Omega), H^1_0(\Omega)\right)}. \tag{26}$$

Convergence rates of sparsity regularization are given in the following theorem.

Theorem 13 *For $q \in \left(\frac{2Q}{Q-2}, \infty\right]$, $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$ and $y \in L^r(\Omega)$. Assume that $\left\|\phi^\delta - \phi^*\right\|_{H^1(\Omega)} \le \delta$ and $\sigma^p_{\alpha,\delta}$ is a solution of (2). Moreover, assume that there exists a function $w^* \in H^{-1}(\Omega)$ such that*

$$\xi := \left(F'_D(\sigma^+)\, y\right)^* (w^*) \in \partial\Phi\left(\sigma^+ - \sigma^0\right). \tag{27}$$

Then,

$$D_\xi\left(\sigma^p_{\alpha,\delta}, \sigma^+\right) = O(\delta) \ \ and \ \ \left\|F_D\left(\sigma^p_{\alpha,\delta}\right) y - \phi^\delta\right\|_{H^1(\Omega)} = O(\delta)$$

as $\delta \to 0$ and $\alpha \sim \delta$. In particular, for $p \in (1, 2]$, we have

$$\left\|\sigma^p_{\alpha,\delta} - \sigma^+\right\|_{L^2(\Omega)} = O\left(\sqrt{\delta}\right).$$

Proof. The proof follows the ideas of Hao and Quyen in [14, 16]. By the definition of $\sigma^p_{\alpha,\delta}$, we get

$$F_{\phi^\delta}\left(\sigma^p_{\alpha,\delta}\right) + \alpha\Phi\left(\sigma^p_{\alpha,\delta} - \sigma^0\right) \le F_{\phi^\delta}\left(\sigma^+\right) + \alpha\Phi\left(\sigma^+ - \sigma^0\right). \tag{28}$$

Then, we have

$$\begin{aligned}
&F_{\phi^\delta}\left(\sigma^p_{\alpha,\delta}\right) + \alpha D_\xi\left(\sigma^p_{\alpha,\delta}, \sigma^+\right)\\
&= F_{\phi^\delta}\left(\sigma^p_{\alpha,\delta}\right) + \alpha\left(\Phi\left(\sigma^p_{\alpha,\delta} - \sigma^0\right) - \Phi\left(\sigma^+ - \sigma^0\right) - \left\langle \xi, \sigma^p_{\alpha,\delta} - \sigma^+\right\rangle_{(L^{q_1}(\Omega'), L^q(\Omega'))}\right)\\
&\le F_{\phi^\delta}\left(\sigma^+\right) - \alpha\left\langle \xi, \sigma^p_{\alpha,\delta} - \sigma^+\right\rangle_{(L^{q_1}(\Omega'), L^q(\Omega'))}\\
&\le \frac{1}{\lambda}\delta^2 - \alpha\left\langle \xi, \sigma^p_{\alpha,\delta} - \sigma^+\right\rangle_{(L^{q_1}(\Omega'), L^q(\Omega'))}.
\end{aligned} \tag{29}$$

From (26) and (27), we get

$$\left\langle \xi, \sigma^p_{\alpha,\delta} - \sigma^+\right\rangle_{(L^{q_1}(\Omega'), L^q(\Omega'))} = \left\langle w^*, F'_D(\sigma^+)\, y\left(\sigma^p_{\alpha,\delta} - \sigma^+\right)\right\rangle_{\left(H^{-1}(\Omega), H^1_0(\Omega)\right)}. \tag{30}$$

By Riesz's representation theorem, there exists an element $w \in H^1_0(\Omega)$ such that

$$\left\langle w^*, F'_D(\sigma^+)\, y\left(\sigma^p_{\alpha,\delta} - \sigma^+\right)\right\rangle_{\left(H^{-1}(\Omega), H^1_0(\Omega)\right)} = \left\langle w, F'_D(\sigma^+)\, y\left(\sigma^p_{\alpha,\delta} - \sigma^+\right)\right\rangle_{H^1_0(\Omega)}. \tag{31}$$

Since $\sigma^+ \ge \lambda > 0$, the scalar product

$$[\phi, v]_{H^1_0(\Omega)} := \int_\Omega \sigma^+ \nabla\phi \cdot \nabla v dx, \text{ for all } \phi, v \in H^1_0(\Omega)$$

is equivalent to $\langle \phi, v\rangle_{H^1_0(\Omega)}$ on $H^1_0(\Omega)$. Therefore, there exists an element $\hat{w} \in H^1_0(\Omega)$ independent of $\sigma^p_{\alpha,\delta}$ such that

$$\left\langle w, F'_D(\sigma^+)\, y\left(\sigma^p_{\alpha,\delta} - \sigma^+\right)\right\rangle_{H^1_0(\Omega)} = \int_\Omega \sigma^+ \nabla\hat{w} \cdot \nabla F'_D(\sigma^+)\, y\left(\sigma^p_{\alpha,\delta} - \sigma^+\right) dx. \tag{32}$$

From (30), (31) and (32), we have

$$\left\langle \xi, \sigma^p_{\alpha,\delta} - \sigma^+\right\rangle_{(L^{q_1}(\Omega'), L^q(\Omega'))} = \int_\Omega \sigma^+ \nabla\hat{w} \cdot \nabla F'_D(\sigma^+)\, y\left(\sigma^p_{\alpha,\delta} - \sigma^+\right) dx =: \Lambda.$$

From the weak solution formulas of $F_D(\sigma^+)\,y$ and $F_D'(\sigma^+)\,y\left(\sigma^p_{\alpha,\delta}-\sigma^+\right)$ (see (6) and (10)), we deduce

$$\begin{aligned}
\alpha\Lambda &= \alpha\int_\Omega \sigma^+\nabla\hat{w}\cdot\nabla F_D'\left(\sigma^+\right)y\left(\sigma^p_{\alpha,\delta}-\sigma^+\right)dx\\
&= -\alpha\int_\Omega\left(\sigma^p_{\alpha,\delta}-\sigma^+\right)\nabla\hat{w}\cdot\nabla F_D\left(\sigma^+\right)ydx\\
&= \alpha\int_\Omega \sigma^+\nabla\hat{w}\cdot\nabla F_D\left(\sigma^+\right)ydx-\alpha\int_\Omega\sigma^p_{\alpha,\delta}\nabla\hat{w}\cdot\nabla F_D\left(\sigma^+\right)ydx\\
&= \alpha\int_\Omega \sigma^p_{\alpha,\delta}\nabla\hat{w}\cdot\nabla F_D\left(\sigma^p_{\alpha,\delta}\right)ydx-\alpha\int_\Omega\sigma^p_{\alpha,\delta}\nabla\hat{w}\cdot\nabla F_D\left(\sigma^+\right)ydx\\
&= \alpha\int_\Omega \sigma^p_{\alpha,\delta}\nabla\hat{w}\cdot\nabla\left(F_D\left(\sigma^p_{\alpha,\delta}\right)y-F_D\left(\sigma^+\right)y\right)dx\\
&= \alpha\int_\Omega \sigma^p_{\alpha,\delta}\nabla\hat{w}\cdot\nabla\left(F_D\left(\sigma^p_{\alpha,\delta}\right)y-\phi^\delta\right)dx+\alpha\int_\Omega\sigma^p_{\alpha,\delta}\nabla\hat{w}\cdot\nabla\left(\phi^\delta-\phi^*\right)dx.
\end{aligned}$$

Using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
\alpha\left|\Lambda\right| &\le \alpha\left(\int_\Omega\sigma^p_{\alpha,\delta}\left|\nabla\hat{w}\right|^2dx\right)^{1/2}\left(\int_\Omega\sigma^p_{\alpha,\delta}\left|\nabla\left(F_D\left(\sigma^p_{\alpha,\delta}\right)y-\phi^\delta\right)\right|^2dx\right)^{1/2}\\
&\quad+\alpha\left(\int_\Omega\left(\sigma^p_{\alpha,\delta}\right)^2\left|\nabla\hat{w}\right|^2dx\right)^{1/2}\left(\int_\Omega\left|\nabla\left(\phi^\delta-\phi^*\right)\right|^2dx\right)^{1/2}\\
&\le \alpha\left(\frac{1}{\lambda}\int_\Omega\left|\nabla\hat{w}\right|^2dx\right)^{1/2}\left(J_{\phi^\delta}\left(\sigma^p_{\alpha,\delta}\right)\right)^{1/2}+\frac{\alpha}{\lambda}\left(\int_\Omega\left|\nabla\hat{w}\right|^2dx\right)^{1/2}\left\|\phi^\delta-\phi^*\right\|_{H^1(\Omega)}\\
&\le \frac{\alpha^2}{2\lambda}\int_\Omega\left|\nabla\hat{w}\right|^2dx+\frac{1}{2}F_{\phi^\delta}\left(\sigma^p_{\alpha,\delta}\right)+\frac{\alpha\delta}{\lambda}\left(\int_\Omega\left|\nabla\hat{w}\right|^2dx\right)^{1/2}. \qquad (33)
\end{aligned}$$

Here, we used the inequality $ab\le\frac{\alpha a^2}{2}+\frac{b^2}{2\alpha}$ for the first term. Together with (29), we deduce

$$\frac{1}{2}F_{\phi^\delta}\left(\sigma^p_{\alpha,\delta}\right)+\alpha D_\xi\left(\sigma^p_{\alpha,\delta},\sigma^+\right)\le\frac{1}{\lambda}\delta^2+\frac{\alpha^2}{2\lambda}C_1^2+\frac{\alpha\delta}{\lambda}C_1, \qquad (34)$$

with $C_1=\left(\int_\Omega\left|\nabla\hat{w}\right|^2dx\right)^{1/2}$. This inequality implies that

$$D_\xi\left(\sigma^p_{\alpha,\delta},\sigma^+\right)=O\left(\delta\right)\ \text{ as }\alpha\to 0\text{ and }\alpha\sim\delta.$$

By (8) and (34), we have

$$\left\|F_D\left(\sigma^p_{\alpha,\delta}\right)y-\phi^\delta\right\|^2_{H^1(\Omega)}\le\frac{1}{C}F_{\phi^\delta}\left(\sigma^p_{\alpha,\delta}\right)=O\left(\delta^2\right)\ \text{ as }\delta\to 0\text{ and }\alpha\sim\delta.$$

In particular, for $p\in(1,2]$ there exists a constant $C_p>0$ such that $D_\xi\left(\sigma^p_{\alpha,\delta},\sigma^+\right)\ge C_p\left\|\sigma^p_{\alpha,\delta}-\sigma^+\right\|^2_{L^2(\Omega)}$, see [13, Lemma 10.]. Therefore, we have

$$\left\|\sigma^p_{\alpha,\delta}-\sigma^+\right\|_{L^2(\Omega)}=O\left(\sqrt{\delta}\right).$$

■

Remark 14 *Our source condition is very simple and is the simplest among the source conditions in [18, 13, 9, 26]. Especially, we do not need the smallness requirement in the source condition.*

5 Conclusion

In this paper, sparsity regularization incorporated with the energy functional approach was analyzed for the diffusion coefficient identification problem. The regularized problem was proven to be well-posed and convergence rates of the method was obtained under a simple source condition. An advantage of the new approach is to work with a convex energy functional. Another advantage is that the source condition of obtaining convergence rates are very simple. We want to emphasize that our source condition is the simplest when it is compared with that in the least squares approach. We did not need the requirement of smallness (or its generalizations) in the source condition.

References

[1] R. Acar and C. R. Vogel. Analysis of bounded variation penalty methods for ill-posed problems. *Inverse Problems*, 10:1217–1229, 1996.

[2] A. Beck and M. Teboulle. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM J. Imaging Sci.*, 2(1):183–202, 2009.

[3] T. Bonesky, K. Bredies, D. A. Lorenz, and P. Maass. A generalized conditional gradient method for nonlinear operator equations with sparsity constraints. *Inverse Problems*, 33:2041–2058, 2007.

[4] K. Bredies, D. A. Lorenz, and P. Maass. A generalized conditional gradient method and its connection to an iterative shrinkage method. *Computational Optimization and Application*, 42(2):173–193, 2009.

[5] T. F. Chan and X. Tai. Identification of discontinuous coefficients in ellptic problems using total variation regularization. *SIAM J. Sci. Comput.*, 25(3):881–904, 2003.

[6] T. F. Chan and X. Tai. Level set and total variation regularization for elliptic inverse problems with discontinuous coefficients. *Journal of Computational Physics*, 193:40–66, 2003.

[7] Z. Chen and J. Zou. An augmented Lagrangian method for identifying discontinuous parameters in elliptic systems. *SIAM Journal of Control and Optimization*, 37:892–910, 1999.

[8] I. Daubechies, M. Defrise, and C. Demol. An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. *Comm. Pure Appl. Math*, 57:1413–1541, 2004.

[9] H. W. Engl, M. Hanke, and A. Neubauer. *Regularization of Inverse Problems*. Kluwer, Dordrecht, 1996.

[10] R. Falk. Error estimates for the numerical identification of a variable coefficient. *Mathematics of Computation*, 40(3):537–546, July 1983.

[11] L. Gan. Block compressed sensing of natural images. In *Digital Signal Processing, 2007 15th International Conference on*, pages 403–406, July 2007.

[12] M. Gehre, T. Kluth, A. Lipponen, B. Jin, A. Seppänen, J. Kaipio, and P. Maass. Sparsity reconstruction in electrical impedance tomography: an experimental evaluation. in press, 2011.

[13] M. Grasmair, M. Haltmeier, and O. Scherer. Sparsity regularization with l^q penalty term. *Inverse Problems*, 24:055020, 2008.

[14] D. N. Hào and T. N. T. Quyen. Convergence rates for Tikhonov regularization of coefficient identification problems in Laplace-type equation. *Inverse Problems*, 26:125014, 2010.

[15] D. N. Hào and T. N. T. Quyen. Convergence rates for Tikhonov regularization of a two-coefficient identification problem in an elliptic boundary value problem. *Numer. Math.*, pages 1–33, 2011.

[16] D. N. Hào and T. N. T. Quyen. Convergence rates for total variation regularization of coefficient identification problems in elliptic equations I. *Inverse Problems*, 27:075008, 2011.

[17] D. N. Hào and T. N. T. Quyen. Convergence rates for total variation regularization of coefficient identification problems in elliptic equations II. Submitted to Journal of Mathematical Analysis and Applications, 2011.

[18] B. Hofmann, B. Kaltenbacher, C. Pöschl, and O. Scherzer. A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators. *Inverse Problems*, 23:987–1010, 2007.

[19] B. Jin and P. Maass. An analysis of electrical impedance tomography with applications to Tikhonov regularization. *http://www.dfg-spp1324.de/download/preprints/preprint070.pdf*, 2011.

[20] Bangti Jin, Taufiquar Khan, and Peter Maass. A reconstruction algorithm for electrical impedance tomography based on sparsity regularization. *International Journal for Numerical Methods in Engineering*, 2011.

[21] I. Knowles. Parameter identification for elliptic problems. *Journal of Computational and Applied Mathematics*, 131:175–194, 2001.

[22] R. V. Kohn and B. Lowe. A variational method for parameter identification. *Mathematical Modeling and Numerical Analysis*, 22:119–158, 1988.

[23] C. Kravaris and J. H. Seinfeld. Identification of parameters in distributed parameter systems by regularization. *SIAM Journal of Control and Optimization*, 23:217–241, 1985.

[24] D. A. Lorenz, P. Maass, and P. Q. Muoi. Gradient descent methods based on quadratic approximations of tikhonov functionals with sparsity constraints: theory and numerical comparison of stepsize rules. preprint, 2011.

[25] N. G. Meyers. An L^p- estimate for the gradient of the solutions of second order elliptic divergence equations. *Ann. Scuola Norm. Sup. Pisa (3)*, 17:189–206, 1963.

[26] A. Neubauer. On enhanced convergence rates for Tikhonov regularization of nonlinear ill-posed problems in Banach spaces. *Inverse Problems*, 25:065009, 1989.

[27] R. Ramlau and G. Teschke. A Tikhonov-based projection iteration for nonlinear ill-posed problems with sparsity constraints. *Numer. Math.*, 104:177–203, 2006.

[28] G. R. Richter. Numerical identification of a spatially varying diffusion coefficient. *Mathematics of Computation*, 36:375–386, 1981.

[29] L. Rondi and F. Santosa. Enhanced electrical impedance tomography via the Mumford-Shah functional. *ESAIM Control Optim. Calc. Var.*, 6:517–538, 2001.

[30] D. Takhar, J. N. Laska, M. B. Wakin, M. F. Duarte, D. Baron, S. Sarvotham, K. F. Kelly, and R. G. Baraniuk. A new compressive imaging camera architecture using optical-domain compression. In *in Proc. of Computational Imaging IV at SPIE Electronic Imaging*, pages 43–52, 2006.

[31] M. B. Wakin, J. N. Laska, M. F. Duarte, D. Baron, S. Sarvotham, D. Takhar, K. F. Kelly, and R. G. Baraniuk. An architecture for compressive imaging. In *IEEE International Conference on Image Processing (ICIP)*, pages 1273–1276, 2006.

[32] L. W. White and J. Zhou. Continuity and uniqueness of regularized output least squares optimal estimators. *Journal of Mathematical Analysis and Applications*, 196:53–83, 1994.

[33] W. Yin, S. Osher, D. Goldfarb, and J. Darbon. Bregman iterative algorithms for ℓ_1-minimization with applications to compressed sensing. *SIAM Journal on Imaging Sciences*, 1(1):143–168, 2008.

[34] J. Zou. Numerical methods for elliptic inverse problems. *International Journal of Computer Mathematics*, 70:211–232, 1998.