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Abstract: Optimal controllers, like the classical Linear Quadratic Regulator closed-loop controller (LQR), have proved to be a powerful tool in many applications and to be robust enough to compensate most differences between simulation and reality. Nevertheless these controllers are not optimal if disturbances or perturbations in the system data occur. If these controllers are applied in a real process, the possibility of data disturbances force recomputing the feedback control law in real-time to preserve stability and optimality, at least approximately. For this purpose, a numerical method based on the parametric sensitivity analysis of nonlinear optimization problems is suggested to calculate higher order approximations of the feedback control law in real-time. Using this method the optimal controller can be adapted within a few nanoseconds on an typical personal computer. The method is illustrated by the adaptive optimal control of the classical inverted pendulum.

Keywords: optimal control, Riccati controller, parametric sensitivity analysis, perturbation

1 Introduction

Optimization and optimal control is a natural and widely used tool for giving a systematic procedure for the design of feedback control systems in modern control theory. In the case of linear systems with full state measurements, the linear quadratic regulator (LQR) problem, also known as the Riccati controller, provides one of the most useful techniques for designing state space controllers. The Riccati controller is known to be locally robust, which is a fundamental requirement in designing feedback control systems. However, this optimal control techniques do not provide the issue of general robustness. Nevertheless robustness properties of the control system reflects an ability of the system to maintain both, adequate performance (optimality) with respect to an user defined objective function and stability in the sense of variations and errors in the model dynamics. Hence the enhancement of robustness is one of the main reasons for using feedback.

This paper is concerned with the numerical solution of more generalized LQR problem. We show that the well known optimal closed-loop controller (state feedback controller) is neither optimal nor robust in the presence of disturbances or perturbations in the system data. Unfortunately, the comparatively high computing times for recalculation the state space controller disqualify commonly used methods. This motivates the development of fast and reliable *real-time optimal approximations* for perturbed optimal controllers.

In stability analysis of finite dimensional nonlinear optimization problems, differentiability properties of optimal solutions with respect to perturbation parameters are studied. Parametric sensitivity analysis is concerned with the computation of parametric sensitivity differentials of optimal solutions. This sensitivity information enables the control engineer to estimate the changes in the optimal feedback law due to deviations of the system matrices from fixed nominal values.

In this paper, we consider an LQR full state feedback law applied to the classical inverted pendulum. An approximate feedback law appropriate for numerical implementation is developed in the context of a fast, first order Taylor expansion of the optimal feedback matrix with respect to system perturbations. This first order approximations leads to a second order approximation of the objective functional and hence defines a near optimal and robust state space controller.

2 Perturbed LQR-Problems

We consider a linear time invariant dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \tag{1}$$

with (for the moment) constant matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$. Herein $x(t) \in \mathbb{R}^n$ denotes the state of the system with initial value $x(0) = x_0$ and $u(t) \in \mathbb{R}^m$ the control input for all $t \in [0, \infty)$. Note, that often a linear differential equation system (1) is obtained by linearization of a nonlinear dynamical systems $\dot{x}(t) = f(x(t), u(t))$. Equation (1) defines a control problem, if we ask for control functions $u(t)$ able to transfer the initial value $x(0) = x_0 \neq 0$ of the state variable to

$$x(t_f) = 0, \qquad t_f \in [0, \infty). \tag{2}$$

The final value $x(t_f) = 0$ is not restrictive, since other values can always be achieved by coordinate transformation. Hereafter we will investigate the infinite time horizon $t_f = \infty$. This is a demanding problem since, on the one hand, a fixed control will not assure, that the state reaches or even converges to zero, and, on the other hand, we can assume that there exist infinitely many control functions satisfying condition (2).

We can take advantage of the second problem by selecting an 'expedient' control function out of the infinite number of possibilities. Therefore we require an objective functional to be minimized:

$$\min_{x,u} F(x, u) := \frac{1}{2}\int_0^\infty x(t)^T Qx(t) + u(t)^T Ru(t)dt, \tag{3}$$

$Q \in \mathbb{R}^{n \times n} \geq 0$ and $R \in \mathbb{R}^{m \times m} > 0$. The LQR approach allows us to choose two sets of parameters, the coefficients of the matrix R and the coefficients of the matrix Q. These matrices will balance the relative importance of the input $u(t)$ and state $x(t)$ in the cost function that we are trying to optimize.
Often an additional output

$$y(t) = Cx(t), \tag{4}$$

is investigated with $y(t) \in \mathbb{R}^r$ and $C \in \mathbb{R}^{r \times n}$. Such output functions come into play due to theoretical aspects, see below. Moreover (4) allows for a description of the relationship between the state $x(t)$ and the measurable output $y(t)$, if e.g. not full state measurement is possible.
In summary, the task is to determine a control function $u : [0, \infty) \to \mathbb{R}^m$ that minimizes the functional (3) subject to the constraints (1), (2) and (4).
The next step in our design process is to assume that we have full state feedback (i.e. that we can measure all states), and find a matrix $K_\infty \in \mathbb{R}^{m \times n}$ which determines the feedback control law

$$u(t) = -K_\infty x(t). \tag{5}$$

This can be done in a number of ways. We use the optimal control theory, which will give us, under certain assumptions, the optimal controller with respect to (3). The following theorem is well known and can be found in the literature, cf. Locatelli [6], Bryson [3], Föllinger [5].

Theorem 2.1 (Optimal Solution of the LQR Problem) *For a given matrix $\bar{Q} \in \mathbb{R}^{r \times r}$ let $Q = C^T \bar{Q} C = D^T D \geq 0$, $D \in \mathbb{R}^{r \times n}$ be a factorization, such that (A, B) is stabilizable and (A, D) is observable. Then there exists one and only one solution $S^* > 0$ of the algebraic Riccati equation*

$$SA - SBR^{-1}B^T S + Q + A^T S = 0, \tag{6}$$

such that

$$u(t) = -K_\infty x(t), \qquad K_\infty := R^{-1} B^T S^* \tag{7}$$

is optimal with respect to the objective F. Moreover (7) defines an asymptotic stable closed-loop system and the optimal value F^* of the objective (3) is given by $F^* = \frac{1}{2} x_0^T S^* x_0 < \infty$.

The assumptions in Theorem 2.1 are more of theoretical importance. For practical reasons the following results from system theory might be helpful:
The system (A, B) is stabilizable, if (A, B) is controllable, if, e.g.

$$rank([B, AB, \ldots, A^{n-1}B]) = n. \tag{8}$$

The system (A, D) is observable, if , e.g.

$$rank \begin{pmatrix} D \\ DA \\ \vdots \\ DA^{n-1} \end{pmatrix} = n. \qquad (9)$$

Hence from a practical point of view only the problem of solving the underlying algebraic Riccati equation (6), which is nonlinear in S, is left. This matrix equation can be solved in numerous ways, e.g. by Newton's method.

Next we extend the linear quadratic regulator problem (1)-(4) to general linear and nonlinear perturbations in the system matrices. In detail we are interested in the perturbed linear quadratic regulator problem

$$\text{(LQR(p))} \quad \begin{array}{lrcl} \min\limits_{x,u} & F(x,u,p) & = & \frac{1}{2}\int_0^\infty x(t)^T Q(p) x(t) + u(t)^T R(p) u(t) dt \\ \text{s.t.} & \dot{x}(t) & = & A(p)x(t) + B(p)u(t) + d(p) \\ & y(t) & = & C(p)x(t) \\ & x(0) & = & x_0(p). \end{array} \qquad (10)$$

Hence we do not have any longer constant matrices but mappings $A : P \longrightarrow \mathbb{R}^{n\times n}$, $B : P \longrightarrow \mathbb{R}^{n\times m}$, $C : P \longrightarrow \mathbb{R}^{r\times n}$, $Q : P \longrightarrow \mathbb{R}^{n\times n}$, $R : P \longrightarrow \mathbb{R}^{m\times m}$, $d : P \longrightarrow \mathbb{R}^n$, $x_0 : P \longrightarrow \mathbb{R}^n$ with $P \subset \mathbb{R}^q$. The feedback law (7) is able only to optimally compensate for perturbations in the initial value $x_0(p)$, but not for perturbations in A, B, C, Q, R or d. Additional constant elements d in the right hand side of the dynamics in (1) could be neglected, due to coordinate transformation reasons. Linear perturbations, e.g. $\triangle A$ with $A(p) := A + \triangle A$, in the system matrices are enclosed in formulation (10), since one can sort the coefficients of e.g. $\triangle A$ into vector p.
If for a fixed value p_0, e.g. $p_0 = 0$, the assumptions in Theorem 2.1 hold for LQR(p_0), we can find an asymptotic stable closed-loop law

$$u(t) = -K_\infty(p_0)x(t), \qquad (11)$$

where $K_\infty(p_0)$ denotes the dependency of the feedback matrix K_∞ on p_0. Hereafter we will investigate situations where on-line perturbations p might occur. This means, that we expect deviations $\triangle p$ from the nominal value p_0, $p = p_0 + \triangle p$, while the practical implementation of the Riccati controller works on an application. We are not able to calculate the optimal feedback law

$$u(t) = -K_\infty(p)x(t) \qquad (12)$$

fast enough, e.g. due to the computational complexity of the solution of the algebraic Riccati matrix equation (6).

Since the optimal feedback law (12) is not available and the (unperturbed) Riccati controller (11) is known to be robust and to be able to compensate for perturbations $p \in U(p_0)$ out of a neighborhood of p_0, the controller (11) tacitly is used, even in the presence of deviations $\triangle p \neq 0$. In this situation one has to accept that in general in the presence of deviations $\triangle p \neq 0$ the controller (11) is not any longer optimal with respect to the objective functional and moreover, that the controller (11) might become instable for larger perturbations, e.g. from outside the neighborhood $U(p_0)$.
Hereafter we will show, how to derive an asymptotic closed-loop law, which is improved in view of optimality and robustness in comparison to (11) and moreover close to but much less time consuming than the optimal feedback law (12).

3 Approximated Feedback Solution

Let p_0 be a nominal perturbation, e.g. $p_0 = 0$. Hereafter let the assumptions of Theorem 2.1 hold for $A(p_0)$, $B(p_0)$, $C(p_0)$, $Q(p_0)$, $R(p_0)$ and $d(p_0) = 0$. Hence the feedback law (11) for the unperturbed problem LQR(p_0) exists and is optimal with respect to the objective functional $F(x, u, p_0)$. Moreover (7) defines an asymptotic stable closed-loop system and we are allowed to replace all $u(t)$ in (10) by (11). Hence LQR(p_0) is reformulated by NLP(p_0), which is given by

$$
(NLP(p)) \quad
\begin{aligned}
\text{Min. } \bar{F}(x, K_\infty, p) &= \frac{1}{2}\int_0^\infty x(t)^T Q(p) x(t) + (K_\infty x(t))^T R(p) K_\infty x(t)\, dt \\
\text{s.t. } \dot{x}(t) &= (A(p) - B(p)K_\infty) x(t) + d(p) \\
x(0) &= x_0(p),
\end{aligned}
\tag{13}
$$

with $\bar{F}(x, K_\infty, p) := F(x, -K_\infty x, p)$, where the output $y(t)$ is neglected. Since the feedback law (11) is optimal and minimizes the objective, it follows, that the optimal solution of (13) minimizes the objective, too.
Note, that (13) is not any longer a perturbed optimal control problem: it is a perturbed finite dimensional nonlinear optimization problem (NLP) with equality constraints. Moreover, due to the theory of initial value problems for linear differential equations systems as in (13), the existence and uniqueness of the state variable x depending on the coefficients of the matrix K_∞ can be assumed. This yields $x(t) = x(t; K_\infty, p)$ and (13) can be further transformed to the unconstrained perturbed nonlinear optimization problem

$$
\begin{aligned}
\text{Min. } \hat{F}(K_\infty, p) = \frac{1}{2}\int_0^\infty & x(t; K_\infty, p)^T Q(p) x(t; K_\infty, p) \\
& + (K_\infty x(t; K_\infty, p))^T R(p) K_\infty x(t; K_\infty, p)\, dt
\end{aligned}
\tag{14}
$$

with $\hat{F}(K_\infty, p) := F(x(t; K_\infty, p), -K_\infty x(t; K_\infty, p), p)$ and $x(t; K_\infty, p)$ is the unique solution of the perturbed linear differential equation system in (13). An optimal solution of (14) solves the feedback law (11) (or (12) respectively), if the assumptions of Theorem 2.1 hold.
Problem (14) defines an NLP problem of the form

$$\min_z H(z, p), \tag{15}$$

which will be investigated hereafter due to reasons of readability. Problem (15) can be solved efficiently for a suitable function $H : \mathbb{R}^{n \cdot m} \times P \longrightarrow \mathbb{R}$ and a fixed parameter $p = p_0$ by standard techniques, e.g. SQP methods. The dimension $n \cdot m$ of the optimization vector z results from the dimension of the matrix K_∞. Unfortunately these methods are neither able to calculate $K_\infty(p)$ on-line in the presence of perturbations and we have to look for additional concepts.

So far we have been able to transform a perturbed optimal control problem into a perturbed NLP problem. We are especially interested in the differentiability of the optimal solution $z(p)$ with respect to the perturbation parameter p. Sufficient conditions for such solution differentiability are given by

Theorem 3.1 (Differentiability of optimal solutions) *Let H be twice continuously differentiable with respect to z and p. Let z_0 be a strong regular local solution of (15) for a fixed parameter p_0:*

1. *(necessary optimality conditions)*

$$\nabla_z H(z_0, p_0) = 0,$$

2. *(second order sufficient conditions)*

$$v^\top \nabla^2_{zz} H(z_0, \eta_0, p_0) v > 0, \forall v \in \mathbb{R}^{n \cdot m}, v \neq 0.$$

There then exists a neighborhood $\mathcal{P}(p_0)$ such that (15) possesses a unique strong regular local solution $z(p)$ for all $p \in \mathcal{P}(p_0)$. Furthermore, $z(p)$ is a continuously differentiable function of p in $\mathcal{P}(p_0)$ and it holds, that

$$\nabla^2_{zz} H(z_0, p_0) \frac{dz}{dp}(p_0) = -\nabla^2_{zp} H(z_0, p_0) \tag{16}$$

Herein $\nabla^2_{zz} H$ denotes the Hessian of the objective. Note, that the left matrix in (16) is non-singular on the assumptions of Theorem 3.1. Hence, the sensitivity differentials dz/dp at p_0 can be calculated explicitly by solving the linear equation system:

$$\frac{dz}{dp}(p_0) = -\left(\nabla^2_{zz} H(z_0, p_0)\right)^{-1} \nabla^2_{zp} H(z_0, p_0) \tag{17}$$

The proof of the theorem is based on the implicit function theorem and can be found for the more general case of constrained nonlinear optimization problems in Fiacco [4] or Büskens [1]. The assumptions in Theorem 3.1 can be checked numerically.

This type of strong C^1-stability of the optimal solution is crucial for designing real-time approximations of perturbed solutions. In general, sensitivity derivatives do not always exist in case of constrained NLP problems, e.g. at points where the set of active constraints changes. However, we deal with an unconstrained problem which allows for some additional features. The sensitivity differentials in (17) permit an approximation of the optimal perturbed solution $z(p)$ by its first order Taylor expansion:

$$z(p) = z(p_0 + \triangle p) \approx \tilde{z}(p) := z(p_0) + \frac{dz}{dp}(p_0)(p - p_0) = z_0 + \frac{dz}{dp}(p_0)\triangle p. \quad (18)$$

The quantities $z(p_0) = z_0$ and $\frac{dz}{dp}(p_0)$ are computed off-line. The benefit of (18) is then, that only a matrix-vector multiplication and a vector-vector addition have to be performed on-line to approximate $z(p)$ very rapidly. Consequently, (14) is particularly suitable for time critical processes and hence can be used as a real-time approximation.

The Sensitivity Theorem 3.1 predicts the existence of a neighborhood where the rank of the controllability and observability matrixes remain unchanged. This guarantees the existence of a perturbed feedback law (12). Hence (18) can be used to improve the feedback law (11) and to approximate the optimal feedback law (12) by

$$u(t) = -\left(K_\infty(p_0) + \frac{dK_\infty}{dp}(p_0)\triangle p\right)x(t), \quad (19)$$

since

$$K_\infty(p) \approx \tilde{K}_\infty(p) = K_\infty(p_0) + \frac{dK_\infty}{dp}(p_0)\triangle p \quad (20)$$

holds.

When dealing with approximations of the form (19) and (20) one has to ensure that a change in the parameter p does not change the structure of the problem, e.g. the stability and observability assumptions in (8) and (9). Hence in order to guarantee a good approximation of the perturbed solution $z(p)$ and respectively $K_\infty(p)$ by (18) the deviation $\triangle p$ must at least not cause a change in the rank of the controllability and observability matrices in (8) and (9). However, the region of possible deviations can be checked off-line by appropriate computational simulations. In the case of perturbations causing structural changes, the space of reasonable perturbations can be covered by a family of nominal values $(p_0)_i$, $i = 1, 2, \ldots$ and sensitivity derivatives to synthesize the perturbed solution adequately. To simplify the subsequent descriptions we assume that deviations $\triangle p$

do not influence the structure an especially the rank of the matrices in (8) and (9) will stay unchanged.

Equations (19) and (20) yield acceptable real-time approximations for at least small perturbations $\triangle p$. To quantify these approximations the following result is of interest, cf. Büskens [2].

Theorem 3.2 *Let the assumptions of Theorem 3.1 hold and let the function H in (15) be three times continuously differentiable with respect to z and p. Then there exists a neighborhood $U(p_0)$ of p_0 with*

$$\begin{aligned} \|z(p) - \tilde{z}(p)\| &= \mathcal{O}(\|\Delta p\|^2), && (21) \\ \|H(z(p), p) - H(\tilde{z}(p), p)\| &= \mathcal{O}(\|\Delta p\|^3). && (22) \end{aligned}$$

Note, that the optimality of the objective is improved in comparison to the first order approximation of the feedback law in (19), although the variables in $\tilde{K}_\infty(p)$ are still of the order of $\mathcal{O}(\|\Delta p\|^2)$.
A feedback law (11), (12) or (19) is instable if the corresponding objective functional tends to infinity if t does. Hence the improved order of optimality (quadratic approximation) in the objective functional for the feedback (19) indicates a more robust feedback law, than one would have expected from the first order Taylor approximation in (20).

Hereafter we illustrate the theoretical results presented before and apply feedback law (19) to the classical inverted pendulum.

4 Example: The balancing rod

To give an example we consider the classical example of a car with a rod on the top, balancing a ball attached to the top of the rod (cf. Figure 1). The equations of motion for the inverse pendulum are given by the well known nonlinear differential equation system:

$$\begin{aligned} (M+m)\ddot{w} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta &= u, \\ (I+ml^2)\ddot{\theta} - mgl\sin\theta &= -ml\ddot{w}\cos\theta. \end{aligned} \qquad (23)$$

Herein the state variable $w : [0,\infty) \longrightarrow \mathbb{R}$ defines the position of the car, while the angle of the pendulum is given by $\theta : [0,\infty) \longrightarrow \mathbb{R}$. The system can be controlled by the acceleration $u : [0,\infty) \longrightarrow \mathbb{R}$. The constant $m = 1[kg]$ denotes the mass of the ball, $M = 10[kg]$ the mass of the car, $l = 1[m]$ the length of the pendulum and $g = 9.80655[m/s^2]$ is the gravitational constant.
Hereafter let perturbations in the system dynamics and in the objective functional be given by $p := (M, m, l, \alpha)^T \in \mathbb{R}^4$ with $p_0 = (10, 1, 1, 0.5)^T$ and α defined in the objective (26),(27).

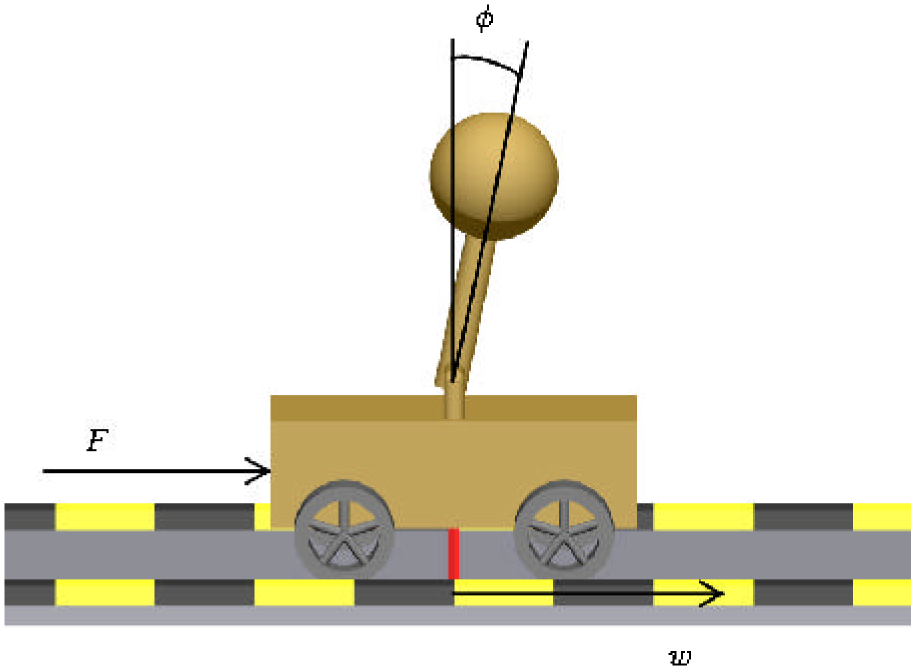

Figure 1: The Balancing Rod

Linearization of (23) around the working point $(w, \dot{w}, \theta, \dot{\theta})^T = (0, 0, 0, 0)^T$ and exploiting $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ leads to the first order linear system

$$\begin{pmatrix} \dot{w}(t) \\ \ddot{w}(t) \\ \dot{\theta}(t) \\ \ddot{\theta}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-0.5gm}{M+0.5m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g(M+m)}{2l(M+0.5m)} & 0 \end{pmatrix}}_{=:A(p)} \begin{pmatrix} w(t) \\ \dot{w}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \frac{1}{M+0.5m} \\ 0 \\ \frac{-1}{2l(M+0.5m)} \end{pmatrix}}_{=:B(p)} u(t), \tag{24}$$

if the mass of the pendulum is assumed to be given as a point mass. For reasons of readability let $x(t) := (w(t), \dot{w}(t), \theta(t), \dot{\theta}(t))^T \in \mathbb{R}^4$, hence (24) is of the form

$$\dot{x}(t) = A(p)x(t) + B(p)u(t), \tag{25}$$

which is equivalent to the definition of the perturbed dynamics in (10).
Note, that the perturbations in (24) and (25) appear non-linear.
The objective functional to be minimized is

$$F(x, u, p) := \frac{1}{2}\int_0^\infty x(t)^T Q(p) x(t) + u(t)^T R(p) u(t)\, dt, \tag{26}$$

with

$$Q(p) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{4\times 4}, \quad R(p) = (1-\alpha) \in \mathbb{R}^{1\times 1} \tag{27}$$

and $\alpha \in]0,1[$ and $\alpha_0 = 0.5$.
Since

$$\begin{aligned} & rank([B(p_0), A(p_0)B(p_0), A(p_0)^2B(p_0), A(p_0)^3B(p_0)]) \\ &= rank \begin{pmatrix} 0.0 & 0.095 & 0.0 & 0.022 \\ 0.095 & 0.0 & 0.022 & 0.0 \\ 0.0 & -0.048 & 0.0 & -0.24 \\ -0.048 & 0.0 & -0.24 & 0.0 \end{pmatrix} = 4 = n \end{aligned} \tag{28}$$

the system is controllable and hence $(A(p_0), B(p_0))$ is stabilizable.
Moreover,

$$D(p_0) := \sqrt{Q(p_0)} = \begin{pmatrix} \sqrt{0.5} & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix} \tag{29}$$

yields the observability of the the system $(A(p_0), D(p_0))$ since

$$rank \begin{pmatrix} D(p_0) \\ D(p_0)A(p_0) \\ \vdots \\ D(p_0)A(p_0)^{n-1} \end{pmatrix} = rank(0.71e_1, 0.71e_5, -0.33e_9, -0.33e_{13}) = 4 = n, \tag{30}$$

where e_i denotes the i'th unit vector.
Hence the assumptions of Theorem 2.1 hold and the unique positive solution $S^* = S^*(p_0)$ of the unperturbed algebraic Riccati equation (6)

$$SA(p_0) - SB(p_0)R(p_0)^{-1}B(p_0)^T S + Q(p_0) + A(p_0)^T S = 0 \tag{31}$$

can be calculated to

$$S^* = S^*(p_0) = \begin{pmatrix} 2.78 & 7.77 & 57.65 & 26.04 \\ 7.77 & 37.44 & 295.34 & 133.41 \\ 57.65 & 295.34 & 7476.63 & 3323.40 \\ 26.04 & 133.41 & 3323.40 & 1477.44 \end{pmatrix} \tag{32}$$

which defines a stable closed-loop system

$$u(t) = -K_\infty(p_0)x(t) \tag{33}$$

in the unperturbed case,

$$K_\infty(p_0) := R(p_0)^{-1}B(p_0)^T S^*(p_0) = -(1.00, 5.57, 260.26, 115.30). \tag{34}$$

Next we consider perturbations $p = (M, m, l, \alpha)^T$. All assumptions of Theorem 3.1 are satisfied for the nominal problem. The Hessian in Theorem 3.1, in particular, is positive definit. There then exists a neighborhood $\mathcal{P}(p_0)$ such that

(24)–(27) possesses a unique strong regular local solution $u(t) = \bar{K}_\infty(p)x(t)$ for all $p \in \mathcal{P}(p_0)$. Furthermore, the matrix $\bar{K}_\infty(p)$ is a continuously differentiable function of p in $\mathcal{P}(p_0)$ and we can calculate the parametric sensitivity differentials $\frac{d\bar{K}_\infty}{dp}(p_0)$ in (17). Since we have $n = 4$, $m = 1$ and $q = 4$ we have a tensor

$$\left(\frac{d\bar{K}_\infty}{dp}(p_0)\right) = \left(\frac{d\bar{K}_\infty}{dp}(p_0)\right)_{\substack{i=1,\dots,m\\ j=1,\dots,n\\ k=1,\dots,q}} \in \mathbb{R}^{m\times n\times q}. \tag{35}$$

To calculate the approximation in (19) we define the matrix $\left(\frac{d\bar{K}_\infty}{dp}(p_0)\right)^{sub} \in \mathbb{R}^{n\times q}$ by

$$\left(\frac{d\bar{K}_\infty}{dp}(p_0)\right)^{sub}_{j,k} := \left(\frac{d\bar{K}_\infty}{dp}(p_0)\right)_{1,j,k}. \tag{36}$$

This yields

$$\left(\frac{d\bar{K}_\infty}{dp}(p_0)\right)^{sub} = \begin{pmatrix} -4.98\cdot 10^{-7} & -4.16\cdot 10^{-7} & -4.70\cdot 10^{-7} & -2.00 \\ -0.21 & -0.20 & -0.44 & -6.46 \\ -21.55 & -20.51 & -24.23 & -48.45 \\ -9.73 & -7.01 & -68.59 & -21.89 \end{pmatrix}, \tag{37}$$

which can be used to rewrite the feedback law in (19) by

$$u(t) = -\left[\bar{K}_\infty(p_0) + \left(\left(\frac{d\bar{K}_\infty}{dp}(p_0)\right)^{sub}\Delta p\right)^T\right] x(t), \tag{38}$$

which will be applied hereafter.

Note, that the sensitivities of the feedback matrix with respect to perturbations in M, m, l for the position of the car are much smaller ($\approx 10^{-7}$) than the others. This indicates, that the value $(\bar{K}_\infty(p))_1$ will be nearly unchanged for perturbations in M, m, l, $(\bar{K}_\infty(p))_1 \approx -1.00$. Moreover, the feedback law in view of the angle of the pendulum is sensitive in case of perturbations, cf. row 3 of (37). Furthermore, as a spin-off, we can note the high sensitivity of the feedback law for the angel velocity in case of perturbations in the length of the pendulum ≈ -68.59.

In order to judge the quality of the real-time feedback adaption for the inverse pendulum, we set up the following deviations

$$\begin{array}{lcllcl}
\Delta p_a &=& (2.0, -0.5, 1.0, 0.0)^T, & \Delta p_b &=& (5.0, 2.0, 1.0, 0.0)^T, \\
\Delta p_c &=& (-0.5, 0.0, 0.0, 0.3)^T, & \Delta p_d &=& (0.0, -0.5, 1.0, 0.0)^T, \\
\Delta p_e &=& (-0.5, -0.5, -0.5, -0.3)^T, & \Delta p_f &=& (3.0, -0.8, 1.0, -0.4)^T, \\
\Delta p_g &=& (-3.0, 1.2, -0.5, 0.1)^T, & \Delta p_h &=& (7.5, 1.5, 1.0, 0.2)^T, \\
\Delta p_i &=& (-2.0, 7.0, 4.0, 0.2)^T, & \Delta p_j &=& (3.0, -0.5, 5.0, -0.1)^T
\end{array}$$

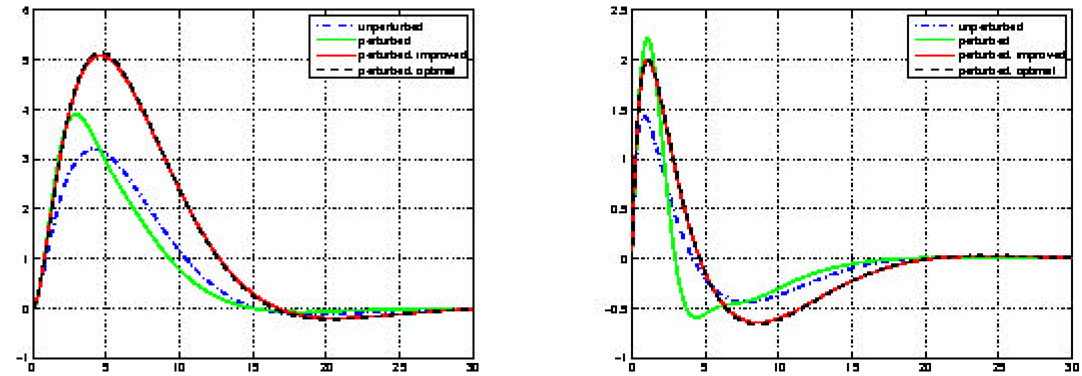

Figure 2: $x_1(t; p_a)$ (left), $x_2(t; p_a)$ (right)

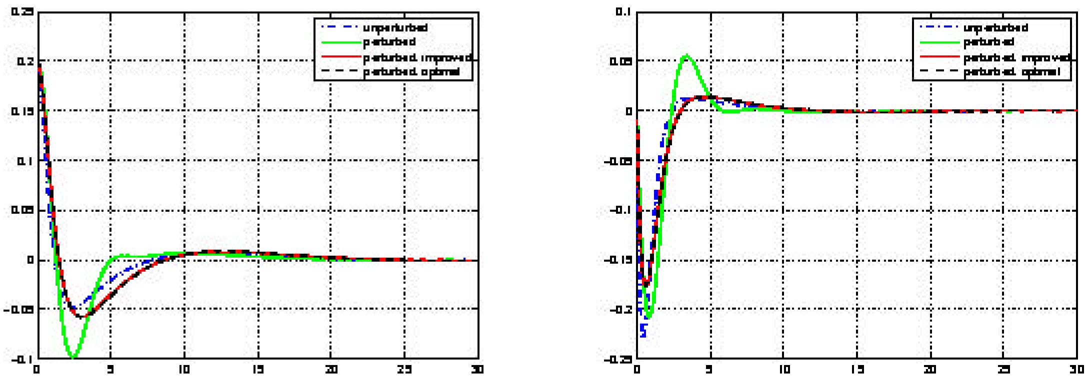

Figure 3: $x_3(t; p_a)$ (left), $x_4(t; p_a)$ (right)

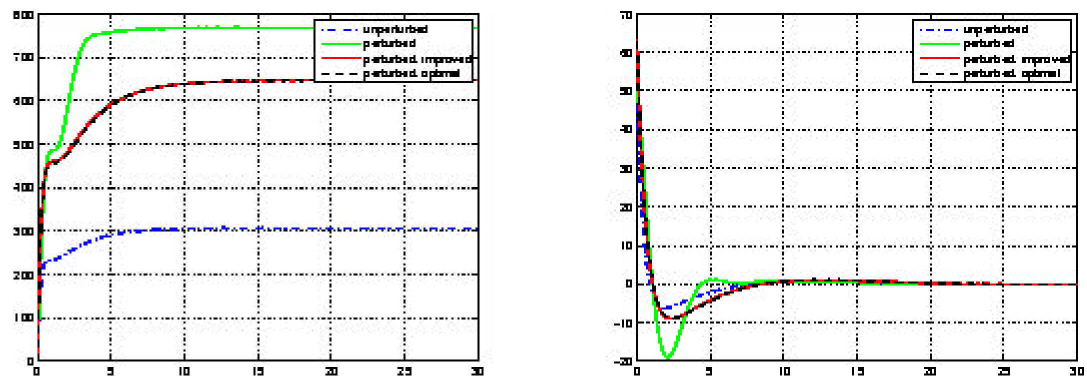

Figure 4: $\int_0^t x_1(\tau; p_a)^2 + u(\tau; p_a)^2 d\tau$ (left), $u(t, p_a)$ (right)

The parameter values are then defined as $p = p_0 + \triangle p_k$, $k \in \{a, \dots, j\}$. To compare the different solutions, we define errors ζ_i, $i = 1, \dots, 4$:

$$\begin{aligned}
\zeta_1 &= \|x_1(t; \tilde{K}(p_0), p) - x_1(t; K(p), p)\|_{\mathcal{L}^2},\\
\zeta_2 &= \|x_1(t; \bar{K}(p), p) - x_1(t; K(p), p)\|_{\mathcal{L}^2},\\
\zeta_3 &= \|u(t; \tilde{K}(p_0), p) - u(t; K(p), p)\|_{\mathcal{L}^2},\\
\zeta_4 &= \|u(t; \bar{K}(p), p) - u(t; K(p), p)\|_{\mathcal{L}^2}.
\end{aligned}$$

Herein

$$\|f\|_{\mathcal{L}^2} = \left(\int_0^{t_f} \|f(t)\|_2 dt \right)^{\frac{1}{2}} \tag{39}$$

denotes the $\mathcal{L}^2$-Norm of a function $f : [0, t_f] \to \mathbb{R}$, $t_f \in \mathbb{R} \cup \{\infty\}$. In extension we present the values of the objective function (26) evaluated for the different feedback solutions, to be able to check the optimality of the solutions with respect to the objective value and to verify the higher order approximation of the objective as predicted in Theorem 3.2.

The feedback laws (11), (12) and (19) obtained for the linear model are used hereafter as closed-loop controllers for the nonlinear dynamics (23) of the inverse pendulum. For all computations the initial value $x(0) = (0, 0, 0.2, 0)^T$ is used.

	$F(K(p_0),p)$	$F(\bar{K}(p),p)$	$F(K(p),p)$	ζ_1	ζ_2	ζ_3	ζ_4
p_a	$7.69 \cdot 10^3$	$6.46 \cdot 10^3$	$6.46 \cdot 10^3$	$2.00 \cdot 10^1$	$7.37 \cdot 10^{-1}$	$4.03 \cdot 10^1$	1.41
p_b	$3.39 \cdot 10^3$	$1.19 \cdot 10^3$	$1.18 \cdot 10^3$	$4.11 \cdot 10^1$	6.43	$2.52 \cdot 10^2$	$1.26 \cdot 10^1$
p_c	$1.48 \cdot 10^3$	$1.33 \cdot 10^3$	$1.32 \cdot 10^3$	$1.35 \cdot 10^1$	3.09	$1.50 \cdot 10^1$	4.25
p_d	$5.40 \cdot 10^3$	$4.77 \cdot 10^3$	$4.76 \cdot 10^3$	$1.22 \cdot 10^1$	2.04	$2.65 \cdot 10^1$	3.04
p_e	$2.53 \cdot 10^3$	$2.44 \cdot 10^3$	$2.43 \cdot 10^3$	8.05	6.31	$1.01 \cdot 10^1$	1.91
p_f	$1.55 \cdot 10^3$	$1.07 \cdot 10^3$	$1.05 \cdot 10^3$	$1.03 \cdot 10^2$	$8.82 \cdot 10^1$	$7.20 \cdot 10^1$	$1.50 \cdot 10^1$
p_g	$1.31 \cdot 10^3$	$1.10 \cdot 10^3$	$1.10 \cdot 10^3$	$1.55 \cdot 10^1$	$5.57 \cdot 10^{-1}$	$1.48 \cdot 10^1$	1.48
p_h	∞	$9.61 \cdot 10^3$	$9.52 \cdot 10^3$	∞	$4.99 \cdot 10^{-3}$	∞	1.18
p_i	∞	$1.18 \cdot 10^3$	$1.18 \cdot 10^3$	∞	0.59	∞	1.13
p_j	∞	$2.04 \cdot 10^3$	$2.01 \cdot 10^3$	∞	1.48	∞	9.16

Table 1: Comparison of Solutions

Case 1: Small perturbations
As a first case we investigate small deviations $\triangle p_a = (2.0, -0.5, 1.0, 0.0)^T$. Figures 2 - 4 show the solutions for the state variables $x_1(t)$-$x_4(t)$, the objective function and the control function $u(t)$ for $t \in [0, 30]$. The dash-dotted curve shows the unperturbed solution for the nominal Riccati controller (11) with unperturbed dynamics, the grey solid curve presents the solution calculated with the nominal Riccati controller (11) but perturbed dynamics, the dashed curve

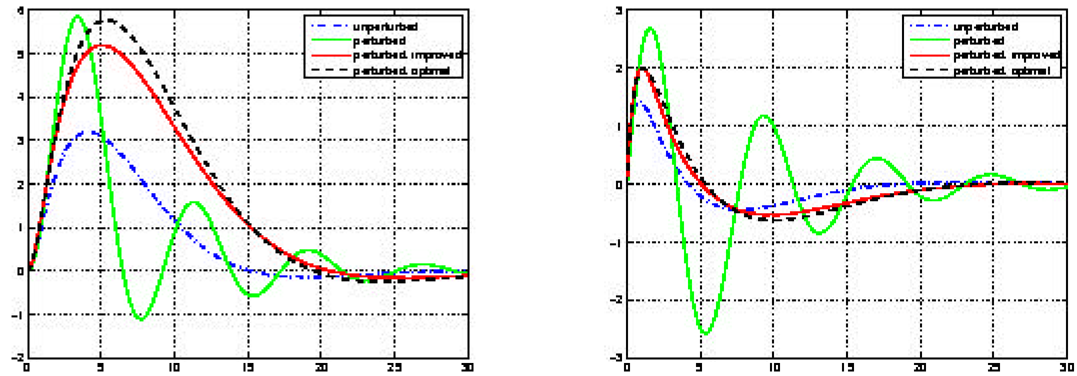

Figure 5: $x_1(t:p_b)$ (left), $x_2(t:p_b)$ (right)

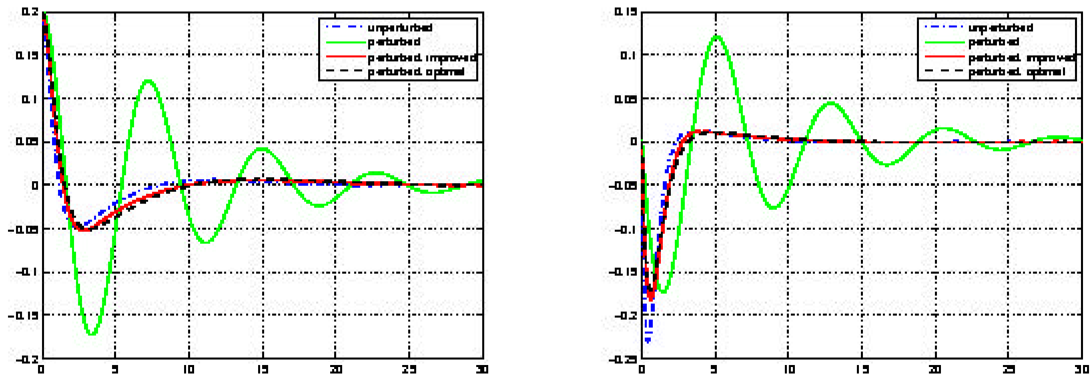

Figure 6: $x_3(t:p_b)$ (left), $x_4(t:p_b)$ (right)

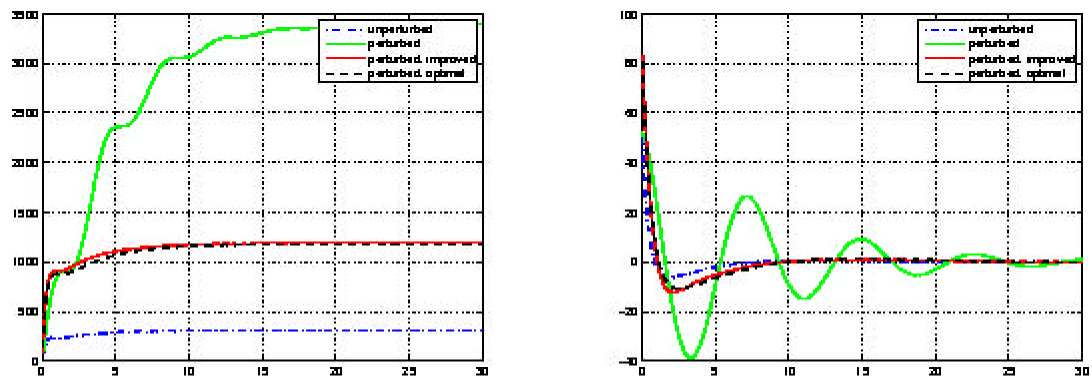

Figure 7: $\int_0^t x_1(\tau:p_b)^2 + u(\tau:p_b)^2 d\tau$ (left), $u(t:p_b)$ (right)

shows the optimal solution of the perturbed problem with Riccati controller (12) and the black solid curve gives the solution of the perturbed problem with the improved optimal controller (19). It should be noted, that the calculation of (19) requires only 16 additional multiplications and additions at each feedback step.
The nominal Riccati controller is able to compensate for these small perturbations in an asymptotic sense, $t \to \infty$. Nevertheless we have to notice, that the classical Riccati controller is far away from the optimal solution. The pictures demonstrate the capacities of the improved controller (19), whose solution is congruent with the exact perturbed solution.

Case 2: Medium perturbations
Next we investigate medium deviations $\triangle p_b = (5.0, 2.0, 1.0, 0.0)^T$. Figures 5 - 7 again present the solutions for state $x_1(t)$-$x_4(t)$, objective and control $u(t)$ for $t \in [0, 30]$. As before we find, that the nominal Riccati controller (11) applied to the perturbed dynamics (grey solid curve) is able to compensate for these medium perturbations in an asymptotic sense, $t \to \infty$. But we observe an unwanted and heavily oscillating behavior of the control variable, which causes oscillations in all states: Position and velocity of the car, angle and angle velocity of the pendulum. The solution obtained by this nominal Riccati controller (11) has nothing in common with the optimal solution obtained by (12) (dashed curve). Elsewise the improved optimal controller (19) applied to the perturbed dynamics (black solid curve): The control and state variables are pretty close to the optimal solution. Except for the position of the car (Figure 5, left). Here we find a somewhat different trajectory for $t \in [3, 13]$, but for $t \in [13, 30]$ the solution is quite congruent with the exact perturbed solution. Note, that the objective is some what different at this time window, but that the objective values are nearly the same at the terminal time $t = 30$, compare Table 1.

Case 3: Larger perturbations
Next we investigate larger deviations $\triangle p_h = (7.5, 1.5, 1.0, 0.2)^T$. Figures 8 - 10 show the solutions for the state and control variables and the objective functional for $t \in [0, 15]$. The nominal Riccati controller (11) applied to the perturbed dynamics (grey solid curve) is not any longer able to compensate for these perturbation not even in an asymptotic sense, $t \to \infty$. The inverted pendulum becomes instable by an increasing oscillating behavior. Hence the objective value $F(K(p_0), p)$ as well as ζ_1, ζ_3 tend to infinity.
Observe, that the improved optimal controller (19) applied to the perturbed dynamics (black solid curve) is not only able to compensate for these perturbation but also very close to the optimal solution obtained by (12) (dashed curve). The optimal value of the objective is achieved very good, compare Table 1, which was predicted by Theorem 3.2. Moreover the robustness properties are demonstrated, which reflect the abilities of the improved optimal controller (19) to maintain ad-

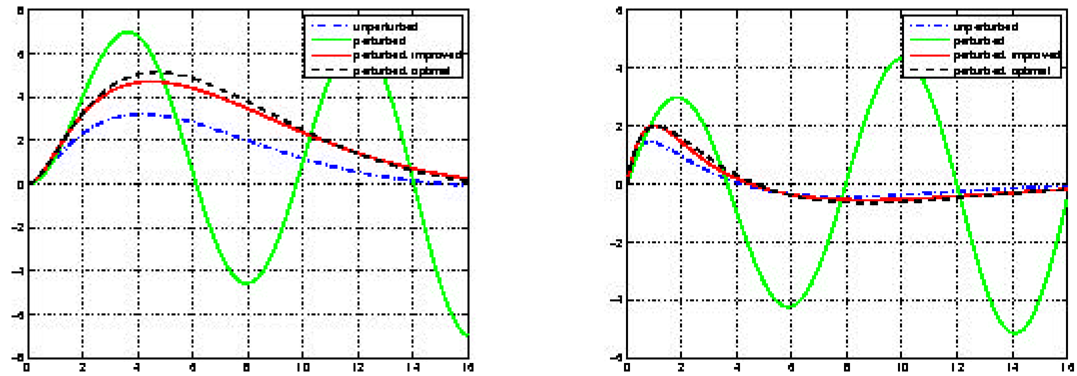

Figure 8: $x_1(t:p_h)$ (left), $x_2(t:p_h)$ (right)

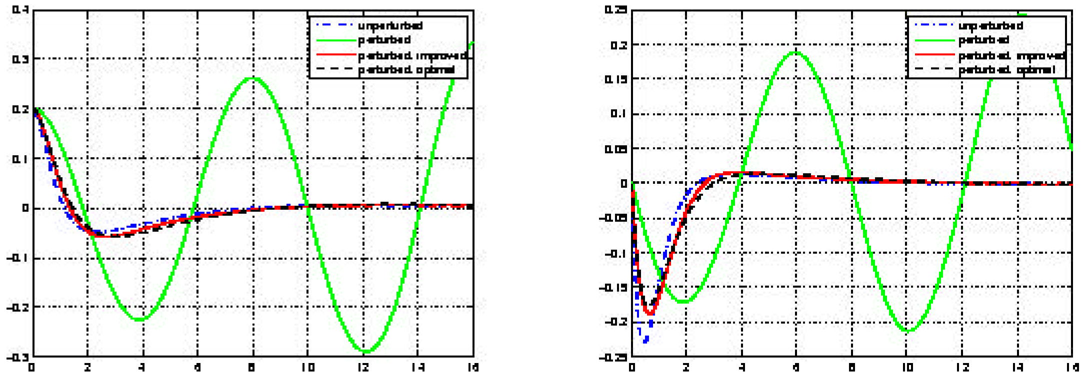

Figure 9: $x_3(t:p_h)$ (left), $x_4(t:p_h)$ (right)

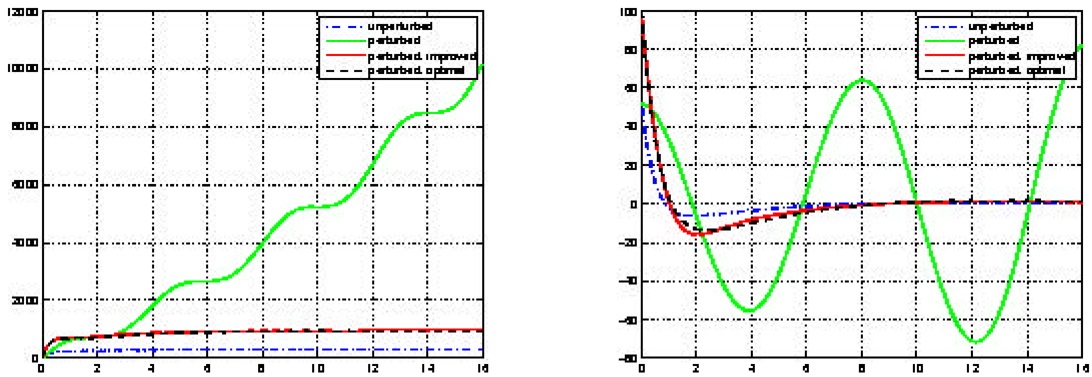

Figure 10: $\int_0^t x_1(\tau:p_h)^2 + u(\tau:p_8)^2 d\tau$ (left), $u(t:p_h)$ (right)

equate performance (optimality) as well as stability in the sense of variations and errors in the model dynamics.

5 Conclusion

The development of robust controllers, able to optimally fulfill an user defined objective function is still a demanding and complex task. This paper is concerned with a new method of calculating higher order approximations of perturbed optimal Riccati controllers (LQR). The technique of parametric sensitivity analysis of unconstrained nonlinear optimization problems was used to improve optimal controllers in the presence of perturbations. It was shown, that the time consuming part of the calculation can be done off-line and an approximation of the perturbed solutions can be given within a few nanoseconds, exploiting the differentiability of the solution using a Taylor expansion. Finally, the capability of the proposed method was shown in the simulation results of an inverted pendulum. The numerical results clearly indicate, that the on-line adaption of the optimal controller approximation exhibit a favorable and robust quality, since the objective is achieved with sufficiently high precision and the computational time for the approximation is much smaller than the recalculation of the exact perturbed optimal controller.

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