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N. Guglielmi, F. Wirth, M. Zennaro

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N. Guglielmi[†], F. Wirth[‡] and M. Zennaro[§]

Abstract

In this paper we consider finite families of complex $n \times n$-matrices. In particular, we focus on those families that satisfy the so-called *Finiteness Conjecture*, which was recently disproved in its more general formulation. We conjecture that the validity of the Finiteness Conjecture for a finite family of *nondefective* type is equivalent to the existence of an *extremal norm* in the class of *complex polytope norms*. However, we have not been able to prove this *Complex Polytope Extremality Conjecture*, but we are able to prove the *Small Complex Polytope Extremality Theorem* under some more restrictive hypotheses on the underlying family of matrices. In addition, our theorem assures a certain finiteness property on the number of vertices of the unit ball of the extremal complex polytope norm, which could be very useful for the construction of suitable algorithms aimed at the actual computation of the *spectral radius* of the family.

Keywords: Families of matrices, joint spectral radius, extremal norms, complex polytope norms, finiteness property.

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[†]Dipartimento di Matematica Pura e Applicata, Università de L'Aquila, I-67100 L'Aquila (Italy), *e-mail:* guglielm@univaq.it

[‡]Zentrum für Technomathematik, Universität Bremen, 28334 Bremen, Germany., *e-mail:* fabian@math.uni-bremen.de

[§]Dipartimento di Scienze Matematiche, Università di Trieste, I-34100 Trieste (Italy), *e-mail:* zennaro@univ.trieste.it

1 Introduction

We consider a bounded family $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$ of complex $n \times n$-matrices, where $\mathcal{I}$ is a set of indices, possibly infinite. For such a family $\mathcal{F}$, the following definitions are given in the literature.

Let $\|\cdot\|$ be a given norm on the vector space $\mathbb{C}^n$ and let the same symbol $\|\cdot\|$ denote also the corresponding induced $n \times n$-matrix norm. Then, for each $k = 0, 1, \ldots$, consider the set $\Sigma_k(\mathcal{F})$ of all possible products of length k whose factors are elements of $\mathcal{F}$, that is

$$\Sigma_k(\mathcal{F}) = \{A^{(i_1)} \ldots A^{(i_k)} \mid i_1, \ldots, i_k \in \mathcal{I}\},$$

with the convention that $\Sigma_0(\mathcal{F}) = \{I\}$, I the identity matrix. Moreover, for each $k = 0, 1, \ldots$, consider the number

$$\hat{\rho}_k(\mathcal{F}) = \sup_{P\in\Sigma_k(\mathcal{F})} \|P\| \tag{1.1}$$

and, finally, define the *joint spectral radius* of $\mathcal{F}$ as

$$\hat{\rho}(\mathcal{F}) = \limsup_{k\to\infty} \hat{\rho}_k(\mathcal{F})^{1/k} \tag{1.2}$$

(see Rota and Strang [RS60]). Note that the numbers $\hat{\rho}_k(\mathcal{F})$ depend on the particular norm $\|\cdot\|$ used in (1.1) whereas, by the equivalence of all the norms in finite dimensional spaces, it turns out that $\hat{\rho}(\mathcal{F})$ is independent of it.

Analogously, let $\rho(\cdot)$ denote the spectral radius of an $n \times n$-matrix and then, for each $k = 0, 1, \ldots$, consider the number

$$\bar{\rho}_k(\mathcal{F}) = \sup_{P\in\Sigma_k(\mathcal{F})} \rho(P)$$

and define the *generalized spectral radius* of $\mathcal{F}$ as

$$\bar{\rho}(\mathcal{F}) = \limsup_{k\to\infty} \bar{\rho}_k(\mathcal{F})^{1/k}$$

(see Daubechies and Lagarias [DL92]). It is not difficult to see (see [DL92]) that

$$\bar{\rho}_k(\mathcal{F}) \le \bar{\rho}(\mathcal{F})^k \le \hat{\rho}(\mathcal{F})^k \le \hat{\rho}_k(\mathcal{F}) \quad \text{for all } k \ge 0 \tag{1.3}$$

and it was later shown that

$$\hat{\rho}(\mathcal{F}) = \bar{\rho}(\mathcal{F})$$

(see Berger and Wang [BW92], Elsner [Els95], Shih *et al.* [SWP97] and Shih [Shi99]). In the light of the above equality, the joint and the generalized spectral radius of $\mathcal{F}$ are the same number, which we shall simply call the *spectral radius* of the (bounded) family of matrices $\mathcal{F}$ and denote by $\rho(\mathcal{F})$.

The above definitions and results are nice generalizations of the well-known situation for *single* families. In particular, the equality $\hat{\rho}(\mathcal{F}) = \bar{\rho}(\mathcal{F})$ is the generalization of the so-called Gelfand limit.

In practical applications, the actual computation of $\rho(\mathcal{F})$ is very important but, unfortunately, if the family $\mathcal{F}$ is not just a single matrix, this is not an easy task at all, see e.g. [Koz90, TB97].

In this paper we try to give a contribution in this direction for special classes of families. More precisely, we find conditions on the family which are sufficient to guarantee the existence of an *extremal norm* in the class of *complex polytope norms*. In addition, under such sufficient conditions, we prove that the unit ball of the extremal norm is a *balanced complex polytope* with a finite *essential system of vertices*. Such a finiteness property is very useful in view of the construction of suitable algorithms aimed at the actual computation and approximation of $\rho(\mathcal{F})$ via the detection of an extremal norm. In Section 2 we review some of the most important definitions and results available in the literature, which are useful for our subsequent developements. In Section 3 we illustrate a few simple results concerning the extremal norms.

In Section 4 we recall the definition of *balanced complex polytope* and of *adjoint balanced complex polytope* and of the corresponding *complex polytope norms* and *adjoint complex polytope norms*, reviewing a few of their most important properties. The main results of the paper are presented in Section 5, where we state the *Complex Polytope Extremality Conjecture*, that is our guess that, for a finite *nondefective* family $\mathcal{F}$, the validity of the Finiteness Conjecture (recently disproved in its more general formulation) be equivalent to the existence of an *extremal norm* in the class of *complex polytope norms*. Indeed, we are able to prove only the *Small Complex Polytope Extremality Theorem* under some more restrictive hypotheses on the underlying family of matrices.

Finally, in Section 6 we present some examples that prove the necessity of the particular hypotheses of the *Small Complex Polytope Extremality Theorem* in order to guarantee some specific finiteness properties for the unit ball of the extremal norm.

Part of the results of this paper have already been successfully applied by Guglielmi and Zennaro [GZ01b] to the analysis of the *zero stability* of some linear multistep methods for the numerical solution of ordinary differential equations. However, at that time even the Small Complex Polytope Extremality Theorem was still at the stage of a conjecture and the mentioned paper [GZ03] makes reference to an early version (in preparation) of the present paper with another title, namely "Polytope norms for families of matrices".

2 Preliminary results from the literature

In this section we briefly review some results from the literature.

In what follows, for the bounded family $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$ of complex $n \times n$-matrices, if $\|\cdot\|$ denotes a norm on the vector space $\mathbb{C}^n$ and the corresponding induced $n \times n$-matrix norm, we shall still use the same notation to define

$$\|\mathcal{F}\| = \hat{\rho}_1(\mathcal{F}) = \sup_{i\in I} \|A^{(i)}\|.$$

The following result can be found, for example, in [RS60] and in Elsner [Els95].

Proposition 2.1 *The spectral radius of a bounded family $\mathcal{F}$ of complex $n \times n$-matrices is characterized by the equality*

$$\rho(\mathcal{F}) = \inf_{\|\cdot\|\in\mathcal{N}} \|\mathcal{F}\|, \qquad (2.1)$$

where $\mathcal{N}$ denotes the set of all possible induced $n \times n$-matrix norms.

Given a family $\mathcal{F}$, an important question to answer is whether or not the inf in (2.1) is actually attained by some induced matrix norm. To this purpose, we give the following definition.

Definition 2.1 *We shall say that a norm $\|\cdot\|_*$ satisfying the condition*

$$\|\mathcal{F}\|_* = \rho(\mathcal{F}) \tag{2.2}$$

is extremal *for the family $\mathcal{F}$.*

It is well known that, for a single family $\{A\}$, the existence of an extremal norm is equivalent to the fact that the matrix A is *nondefective*, i.e., all of the blocks relevant to the eigenvalues of maximum modulus are diagonal in its Jordan canonical form. Whenever $\rho(A) > 0$, another equivalent property is that, with $\hat{A} = \rho(A)^{-1}A$, the power set $\Sigma(\hat{A}) = \{\hat{A}^k \mid k \geq 1\}$ is bounded. These results generalize to a bounded family $\mathcal{F}$ as follows. Given a bounded family $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$ of complex $n\times n$-matrices with $\rho(\mathcal{F}) > 0$, let us consider the *normalized* family

$$\hat{\mathcal{F}} = \{\rho(\mathcal{F})^{-1}A^{(i)}\}_{i\in\mathcal{I}},$$

whose spectral radius is $\rho(\hat{\mathcal{F}}) = 1$. Then consider the semigroup of matrices generated by $\hat{\mathcal{F}}$, i.e.

$$\Sigma(\hat{\mathcal{F}}) = \bigcup_{k\geq 1} \Sigma_k(\hat{\mathcal{F}}).$$

Definition 2.2 *A bounded family $\mathcal{F}$ of complex $n \times n$-matrices is said to be* defective *if the corresponding normalized family $\hat{\mathcal{F}}$ is such that the semigroup $\Sigma(\hat{\mathcal{F}})$ is an unbounded set of matrices. Otherwise, if $\Sigma(\hat{\mathcal{F}})$ is bounded, then the family $\mathcal{F}$ is said to be* nondefective.

Note that we gave the definition of defective family without involving directly the spectral properties of its elements. The following result can be found, for example, in [Koz90] or [BW92].

Proposition 2.2 *A bounded family $\mathcal{F}$ of complex $n \times n$-matrices admits an extremal norm $\|\cdot\|_*$ if and only if it is nondefective. Moreover, if $\mathcal{F}$ is nondefective, any given norm $\|\cdot\|$ on the space of vectors $x \in \mathbb{C}^n$ determines the extremal norm*

$$\|x\|_* = \sup_{k\geq 0} \sup_{P\in\Sigma_k(\mathcal{F})} \frac{\|Px\|}{\rho(\mathcal{F})^k}. \tag{2.3}$$

From Proposition 2.2 it turns out that, for a nondefective family, each vector norm $\|\cdot\|$ canonically determines an extremal norm. However, it is worth remarking that, although (2.3) gives a constructive way of finding an extremal norm, its importance is mainly theoretical since it is often useless from a practical point of view. In order to state another important result about defective and nondefective families (see Barabanov [Bar88] or Elsner [Els95]), we give the following definition according to [RR00].

Definition 2.3 *A bounded family $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$ of complex $n \times n$-matrices is said to be* reducible *if there exist a nonsingular $n\times n$-matrix M and two integers $n_1, n_2 \geq 1$, $n_1+n_2 = n$, such that, for all $i \in \mathcal{I}$, it holds that*

$$M^{-1}A^{(i)}M = \begin{bmatrix} A_{11}^{(i)} & A_{12}^{(i)} \\ O & A_{22}^{(i)} \end{bmatrix},$$

where the blocks $A_{11}^{(i)}$, $A_{12}^{(i)}$, $A_{22}^{(i)}$ are $n_1 \times n_1$-, $n_1 \times n_2$- and $n_2 \times n_2$-matrices, respectively. If a family $\mathcal{F}$ is not reducible, then it is said to be irreducible.

Theorem 2.1 *If a bounded family $\mathcal{F}$ of complex $n \times n$-matrices is defective, then it is reducible.*

Note that irreducibility is a generic property. Furthermore, in a suitable neighborhood of an irreducible family $\mathcal{F}_0$ the spectral radius is a Lipschitz continuous function of $\mathcal{F}$ (see Wirth [Wir02]). We remark that, whereas a defective family is always reducible, the opposite implication is not necessarily true. For example, for $n \geq 2$ all single families $\{A\}$ are clearly reducible, but not necessarily defective. See again [Wir02] for conditions that ensure nondefectiveness for reducible sets $\mathcal{F}$. The following corollary to Theorem 2.1 is obvious.

Corollary 2.1 *If a bounded family $\mathcal{F}$ of complex $n \times n$-matrices is irreducible, then it is nondefective.*

We conclude this section by recalling an important conjecture, arisen from work of Daubechies and Lagarias [DL92] and stated by Lagarias and Wang [LW95], whose validity would be of much help for the actual computation of the spectral radius $\rho(\mathcal{F})$ of finite families.

Definition 2.4 (Finiteness Property) *A finite family of complex $n \times n$-matrices $\mathcal{F}$ is said to have the* finiteness property *if, there exist $k^* \geq 1$ and a product $\bar{P} \in \Sigma_{k^*}(\mathcal{F})$ such that*

$$\rho(\mathcal{F}) = \bar{\rho}_{k^*}(\mathcal{F})^{1/k^*} = \rho(\bar{P})^{1/k^*}. \tag{2.4}$$

Definition 2.5 *If $\mathcal{F}$ is a bounded family of complex $n \times n$-matrices, any matrix $\bar{P} \in \Sigma_{k^*}(\mathcal{F})$ satisfying* (2.4) *for some $k^* \geq 1$ will be called a* spectrum-maximizing product *(in short, an* s.m.p.*) for $\mathcal{F}$.*

Observe that the finiteness property yields the existence of at least one s.m.p. $\bar{P}$ for finite families. Lagarias and Wang were able to give sufficient conditions in terms of extremal norms guaranteeing that the finiteness property holds. The general *Finiteness Conjecture* stating that all finite sets of matrices have the finiteness property was instead recently disproved by Bousch and Mairesse [BM02] and, later, by Blondel *et al.* [BTV03]. Now we recall the following definition from [GZ01a].

Definition 2.6 *Assume that $\mathcal{F}$ is a normalized bounded family of complex $n \times n$-matrices (i.e., $\rho(\mathcal{F}) = 1$) and that there exists a sequence of products $P_k \in \Sigma_{d_k}(\mathcal{F})$, d_k nondecreasing integers, such that*

$$\lim_{k \to \infty} P_k = \tilde{P}, \tag{2.5}$$

where $\tilde{P} \in \overline{\Sigma(\mathcal{F})}$ and $\rho(\tilde{P}) = 1$. Then $\tilde{P}$ will be called a limit spectrum-maximizing product *(in short, an* l.s.m.p.*) for $\mathcal{F}$.*

Note that, for a normalized family $\mathcal{F}$, an s.m.p. $\bar{P}$ is an l.s.m.p., too. To see this, just put $P_k = \bar{P}$ for all $k \geq 1$. Moreover, if the family $\mathcal{F}$ is nondefective, another possibility is to consider the power sequence $\{\bar{P}^k\}_{k\geq 1}$ and, since $\Sigma(\mathcal{F})$ is bounded, to extract a subsequence $\{\bar{P}^{k_s}\}_{s\geq 1}$ converging to some $\tilde{P} \in \overline{\Sigma(\mathcal{F})}$, which obviously satisfies $\rho(\tilde{P}) = 1$. For the sake of brevity, we shall say that such a limit point of the sequence $\{\bar{P}^k\}_{k\geq 1}$ is an *infinite power* of the matrix $\bar{P}$. For nondefective families, Guglielmi and Zennaro [GZ03] proved the following result.

Theorem 2.2 *Let $\mathcal{F}$ be a (possibly infinite) nondefective bounded family of complex $n \times n$-matrices. Then there exists an* l.s.m.p. $\tilde{P}$ *for the normalized family* $\hat{\mathcal{F}}$.

On the contrary, for defective families they gave some counterexamples to the existence of l.s.m.p.'s whenever the dimension of the matrices is $n \geq 4$.

3 Some properties of extremal norms

In this section we consider nondefective bounded families $\mathcal{F}$ of complex $n \times n$-matrices and find out some straightforward properties of the extremal norms. Here $\|\cdot\|_*$ denotes an extremal norm for $\mathcal{F}$. Note first that as an easy consequence of Definition 2.1 we have

$$\|P\|_* \leq \rho(\mathcal{F})^k \quad \text{for all} \quad P \in \Sigma_k(\mathcal{F}). \tag{3.1}$$

Using (3.1) and the submultiplicative property of the induced matrix norms we immediately see that if $P \in \Sigma_k(\mathcal{F})$ and $Q \in \Sigma_h(\mathcal{F})$ are such that $\|PQ\|_* = \rho(\mathcal{F})^{k+h}$, then they satisfy the equalities

$$\|P\|_* = \rho(\mathcal{F})^k \quad \text{and} \quad \|Q\|_* = \rho(\mathcal{F})^h. \tag{3.2}$$

The next statement is obtained just by iterating (3.2).

Proposition 3.1 *Let $\mathcal{F} = \{A^{(i)}\}_{i \in \mathcal{I}}$ be a nondefective bounded family of complex $n \times n$-matrices and let $\|\cdot\|_*$ be an extremal norm for $\mathcal{F}$. If $P = A^{(i_k)} \ldots A^{(i_1)} \in \Sigma_k(\mathcal{F})$ satisfies $\|P\|_* = \rho(\mathcal{F})^k$, then it holds that $\|A^{(i_r)}\|_* = \rho(\mathcal{F})$ for all factors $A^{(i_r)}$, $r = 1, \ldots, k$, of P.*

Lemma 3.1 *Let $\mathcal{F}$ be a nondefective bounded family of complex $n \times n$-matrices and let $\|\cdot\|_*$ be an extremal norm for $\mathcal{F}$. If $\bar{P} \in \Sigma_{k^*}(\mathcal{F})$ is an s.m.p. for $\mathcal{F}$, then*

$$\|\bar{P}\|_* = \rho(\mathcal{F})^{k^*}. \tag{3.3}$$

Proof. This is an easy consequence of (2.4) and (3.1). ■

Definition 3.1 *Let A be a complex $n \times n$-matrix and let $\|\cdot\|$ be a norm on $\mathbb{C}^n$. Then any vector $x \in \mathbb{C}^n$, $x \neq 0$, such that $\|Ax\| = \|A\| \cdot \|x\|$ will be said to be* maximizing *for A with respect to the norm $\|\cdot\|$.*

Again, (3.1) and the submultiplicative property of the induced matrix norms yield immediately the following result.

Proposition 3.2 *Let $\mathcal{F}$ be a nondefective bounded family of complex $n \times n$-matrices and let $\|\cdot\|_*$ be an extremal norm for $\mathcal{F}$. Moreover, let $P \in \Sigma_k(\mathcal{F})$ and $Q \in \Sigma_h(\mathcal{F})$ be such that $\|PQ\|_* = \rho(\mathcal{F})^{k+h}$. If $x \in \mathbb{C}^n$ is maximizing for PQ with respect to the norm $\|\cdot\|_*$, then x is maximizing also for Q and Qx is maximizing for P.*

We conclude this section with a theorem which will be useful in Section 5.

Theorem 3.1 *Assume that a nondefective bounded family $\mathcal{F}$ of complex $n \times n$-matrices has an s.m.p. $\bar{P} = A^{(i_{k^*})} \dots A^{(i_1)}$ and let $\| \cdot \|_*$ be an extremal norm. Then, if $x \neq 0$ is an eigenvector of $\bar{P}$ corresponding to an eigenvalue λ with $|\lambda| = \rho(\bar{P}) = \rho(\mathcal{F})^{k^*}$, then it holds that $\|P^{(r)}x\|_* = \rho(\mathcal{F})^r \|x\|_*$ for all right factors $P^{(r)} = A^{(i_r)} \dots A^{(i_1)}$ of the s.m.p. $\bar{P}$, $r = 1, \dots, k^*$.*

Proof. Observe that, by (3.3), the eigenvector x satisfies the equalities $\|\bar{P}x\|_* = \|\lambda x\|_* = \rho(\mathcal{F})^{k^*}\|x\|_* = \|\bar{P}\|_*\|x\|_*$, i.e. x is maximizing for $\bar{P}$ with respect to the norm $\| \cdot \|_*$. Then apply Proposition 3.2 iteratively, taking (3.2) into account. ■

4 Complex polytopes and related norms

In this section we define *complex polytopes* as generalizations of real polytopes (see, e.g. Ziegler [Zie95]) to the complex case and, consequently, we extend the concept of *polytope norm* to the complex case in a straightforward way. Most of the results (here given without proof) are just either more particular or more general instances of other results which can be found in the literature. In any case, a detailed and self-contained presentation of this topic may be found in Guglielmi and Zennaro [GZ04].

Definition 4.1 *We shall say that a set $\mathcal{X} \subset \mathbb{C}^n$ is* absolutely convex *if, for all $x', x'' \in \mathcal{X}$ and $\lambda', \lambda'' \in \mathbb{C}$ such that $|\lambda'| + |\lambda''| \leq 1$, it holds that $\lambda' x' + \lambda'' x'' \in \mathcal{X}$.*

Definition 4.2 *Let $\mathcal{X} \subset \mathbb{C}^n$. Then the intersection of all absolutely convex sets containing $\mathcal{X}$ will be called the* absolutely convex hull *of $\mathcal{X}$ and will be denoted by* $\mathrm{absco}(\mathcal{X})$.

It is well known that $\mathrm{absco}(\mathcal{X})$ is the set of all the finite absolutely convex linear combinations of vectors of $\mathcal{X}$, i.e. $x \in \mathrm{absco}(\mathcal{X})$ if and only if there exist $x^{(1)}, \dots, x^{(k)} \in \mathcal{X}$ with $k \geq 1$ such that

$$x = \sum_{i=1}^{k} \lambda_i x^{(i)} \quad \text{with} \quad \lambda_i \in \mathbb{C} \quad \text{and} \quad \sum_{i=1}^{k} |\lambda_i| \leq 1.$$

In particular, if $\mathcal{X} = \{x^{(i)}\}_{1 \leq i \leq m}$ is a finite set of vectors, then

$$\mathrm{absco}(\mathcal{X}) = \Big\{ x \in \mathbb{C}^n \;\Big|\; x = \sum_{i=1}^{m} \lambda_i x^{(i)} \text{ with } \lambda_i \in \mathbb{C} \text{ and } \sum_{i=1}^{m} |\lambda_i| \leq 1 \Big\} \tag{4.1}$$

and, in this case, it is a closed subset of $\mathbb{C}^n$. In the sequel, if $\mathcal{X}'$ and $\mathcal{X}''$ are two subsets of $\mathbb{C}^n$, we shall write $\mathcal{X}' \subset \mathcal{X}''$ ($\mathcal{X}' \supset \mathcal{X}''$) to denote proper inclusions, i.e., $\mathcal{X}' \subseteq \mathcal{X}''$ ($\mathcal{X}' \supseteq \mathcal{X}''$) and $\mathcal{X}' \neq \mathcal{X}''$. The forthcoming definition extends the usual definition of symmetric polytope in the real space $\mathbb{R}^n$.

Definition 4.3 *We shall say that a bounded set $\mathcal{P} \subset \mathbb{C}^n$ is a* balanced complex polytope *(in short, a b.c.p.) if there exists a finite set $\mathcal{X} = \{x^{(i)}\}_{1 \leq i \leq m}$ of vectors such that* $\mathrm{span}(\mathcal{X}) = \mathbb{C}^n$ *and*

$$\mathcal{P} = \mathrm{absco}(\mathcal{X}). \tag{4.2}$$

Moreover, if $\mathrm{absco}(\mathcal{X}') \subset \mathrm{absco}(\mathcal{X})$ *for all $\mathcal{X}' \subset \mathcal{X}$, then $\mathcal{X}$ will be called an* essential system of vertices *for $\mathcal{P}$, whereas any vector $ux^{(i)}$ with $u \in \mathbb{C}$, $|u| = 1$, will be called a* vertex *of $\mathcal{P}$.*

Remark that, geometrically speaking, a b.c.p. $\mathcal{P}$ is not a classical polytope. In fact, if we identify the complex space $\mathbb{C}^n$ with the real space $\mathbb{R}^{2n}$, we can easily see that $\mathcal{P}$ is not bounded by hyperplanes. In general, even the intersection $\mathcal{P} \bigcap \mathbb{R}^n$ is not a classical polytope. However, if the b.c.p. $\mathcal{P}$ admits an essential system of real vertices, then $\mathcal{P} \bigcap \mathbb{R}^n$ is a classical polytope. In order to recall the concept of *adjoint* set (that, in the literature, is often referred to as *polarity* or *duality* (see again [Zie95] or Heuser [Heu82]), we consider the usual Euclidean scalar product in $\mathbb{C}^n$ defined by $< x, y >= \sum_{j=1}^n x_j \bar{y}_j$.

Definition 4.4 *Let $\mathcal{X} \subset \mathbb{C}^n$. Then the set*

$$\operatorname{adj}(\mathcal{X}) = \left\{ y \in \mathbb{C}^n \;\Big|\; | < y, x > | \leq 1 \text{ for all } x \in \mathcal{X} \right\} \tag{4.3}$$

will be called the adjoint *of $\mathcal{X}$.*

It is immediately seen that $\operatorname{adj}(\mathcal{X})$ is closed and absolutely convex.

Definition 4.5 *We shall say that a bounded set $\mathcal{P}^* \subset \mathbb{C}^n$ is a b.c.p. of* adjoint *type (in short, an a.b.c.p.) if there exists a finite set $\mathcal{X} = \{x^{(i)}\}_{1 \leq i \leq m}$ of vectors such that* $\operatorname{span}(\mathcal{X}) = \mathbb{C}^n$ *and*

$$\mathcal{P}^* = \operatorname{adj}(\mathcal{X}) = \left\{ y \in \mathbb{C}^n \;\Big|\; | < y, x^{(i)} > | \leq 1,\; i = 1, \ldots, m \right\}. \tag{4.4}$$

Moreover, if $\operatorname{adj}(\mathcal{X}') \supset \operatorname{adj}(\mathcal{X})$ *for all $\mathcal{X}' \subset \mathcal{X}$, then $\mathcal{X}$ will be called an* essential system of facets *for $\mathcal{P}^*$, whereas any vector $ux^{(i)}$ with $u \in \mathbb{C}$, $|u| = 1$, will be called a* facet *of $\mathcal{P}^*$.*

Unlike the case of classical polytopes in $\mathbb{R}^n$, it is not true that the class of b.c.p.'s coincides with the class of a.b.c.p.'s. Indeed, for every b.c.p. $\mathcal{P}$, the equality $\mathcal{P} = \operatorname{adj}(\mathcal{X})$ implies that $\mathcal{X}$ is an infinite set of vectors and, analogously, the same implication holds whenever we express an a.b.c.p. $\mathcal{P}^*$ in the form $\mathcal{P}^* = \operatorname{absco}(\mathcal{X})$. In fact, although the essential system of vertices (facets) $\mathcal{X}$ of a b.c.p. $\mathcal{P}$ (of an a.b.c.p. $\mathcal{P}^*$) is finite, the total number of vertices (facets) is infinite. An interesting geometric property of the boundary of an a.b.c.p. $\mathcal{P}^*$ is that it is *piecewise algebraic* in the following sense.

Proposition 4.1 *Let $\mathcal{P}^*$ be an a.b.c.p. and $\mathcal{X} = \{x^{(i)}\}_{1 \leq i \leq m}$ be an essential system of facets for $\mathcal{P}^*$. Moreover, let $\Phi : \mathbb{C}^n \longrightarrow \mathbb{R}^{2n}$ be the standard vector space isomorphism such that $\Phi(x) = \begin{bmatrix} \Re(x) \\ \Im(x) \end{bmatrix}$ for all $x = \Re(x) + \mathrm{i}\Im(x) \in \mathbb{C}^n$. Then the boundary of $\Phi(\mathcal{P}^*)$ is contained in the zero set of a polynomial $p(z_1, \ldots, z_{2n}) \in \mathbb{R}[z_1, \ldots, z_{2n}]$ of degree $2m$ such that $p(0, \ldots, 0) \neq 0$.*

One of the most important relationships between b.c.p.'s and a.b.c.p.'s is given by the following result that, in the literature, is often referred to as the *bipolar theorem.*

Theorem 4.1 *Let $\mathcal{P}$ be a b.c.p. and let $\mathcal{P}^* = \operatorname{adj}(\mathcal{P})$. Then it holds that*

$$\mathcal{P} = \operatorname{adj}(\mathcal{P}^*) = \operatorname{adj}(\operatorname{adj}(\mathcal{P})). \tag{4.5}$$

Conversely, let $\mathcal{P}^*$ be an a.b.c.p. and let $\mathcal{P} = \operatorname{adj}(\mathcal{P}^*)$. Then it holds that

$$\mathcal{P}^* = \operatorname{adj}(\mathcal{P}) = \operatorname{adj}(\operatorname{adj}(\mathcal{P}^*)). \tag{4.6}$$

The next two propositions state that, given a b.c.p. $\mathcal{P}$ (an a.b.c.p. $\mathcal{P}^*$), the essential system of vertices (facets) $\mathcal{X}$ is uniquely determined modulo scalar factors of modulus equal to 1.

Proposition 4.2 *Assume that $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ and $\bar{\mathcal{X}} = \{\bar{x}^{(i)}\}_{1\le i\le k}$ are two essential systems of vertices for a b.c.p. $\mathcal{P}$. Then $k = m$ and, for each $i = 1,\dots,m$, there exist j_i, $1 \le j_i \le m$, and $u_i \in \mathbb{C}$, $|u_i| = 1$, such that $\bar{x}^{(i)} = u_i x^{(j_i)}$.*

Proposition 4.3 *Assume that $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ and $\bar{\mathcal{X}} = \{\bar{x}^{(i)}\}_{1\le i\le k}$ are two essential systems of facets for an a.b.c.p. $\mathcal{P}^*$. Then $k = m$ and, for each $i = 1,\dots,m$, there exist j_i, $1 \le j_i \le m$, and $u_i \in \mathbb{C}$, $|u_i| = 1$, such that $\bar{x}^{(i)} = u_i x^{(j_i)}$.*

Now we extend the concept of *polytope norm* to the complex case in a straightforward way.

Lemma 4.1 *Any b.c.p. $\mathcal{P}$ is the unit ball of a norm $\|\cdot\|_{\mathcal{P}}$ on $\mathbb{C}^n$.*

Definition 4.6 *We shall call* complex polytope norm *any norm $\|\cdot\|_{\mathcal{P}}$ whose unit ball is a b.c.p. $\mathcal{P}$.*

Lemma 4.2 *Any a.b.c.p. $\mathcal{P}^*$ is the unit ball of a norm $\|\cdot\|_{\mathcal{P}^*}$ on $\mathbb{C}^n$.*

Definition 4.7 *We shall call* adjoint complex polytope norm *any norm $\|\cdot\|_{\mathcal{P}^*}$ whose unit ball is an a.b.c.p. $\mathcal{P}^*$.*

An important link between polytope norms and adjoint polytope norms is illustrated by the following theorem.

Theorem 4.2 *Let $\mathcal{P}$ be a b.c.p. and let $\|\cdot\|_{\mathcal{P}}$ be the corresponding complex polytope norm. Then, for any $z \in \mathbb{C}^n$, it holds that*

$$\|z\|_{\mathcal{P}} = \min\Big\{ \sum_{i=1}^{m} |\lambda_i| \;\Big|\; z = \sum_{i=1}^{m} \lambda_i x^{(i)} \Big\} = \max_{y\in\partial\mathcal{P}^*} |< z, y >|, \tag{4.7}$$

where $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$ and $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ is an essential system of vertices for $\mathcal{P}$. Analogously, let $\mathcal{P}^*$ be an a.b.c.p. and let $\|\cdot\|_{\mathcal{P}^*}$ the corresponding adjoint complex polytope norm. Then, for any $z \in \mathbb{C}^n$, it holds that

$$\|z\|_{\mathcal{P}^*} = \max_{1\le i\le m} |< z, x^{(i)} >| = \max_{x\in\partial\mathcal{P}} |< z, x >|, \tag{4.8}$$

where $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$ and $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ is an essential system of facets for $\mathcal{P}^*$.

Corollary 4.1 *Let $\mathcal{P}$ be a b.c.p. and let $\|\cdot\|_{\mathcal{P}}$ the corresponding complex polytope norm. Moreover, let $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$ and let $\|\cdot\|_{\mathcal{P}^*}$ the corresponding adjoint complex polytope norm. Then, for any complex $n \times n$-matrix A and its adjoint A^*, it holds that*

$$\|A^*\|_{\mathcal{P}^*} = \|A\|_{\mathcal{P}}. \tag{4.9}$$

The next theorem shows that the set of the complex polytope norms is *dense* in the set of all norms defined on $\mathbb{C}^n$ and that, consequently, the corresponding set of induced matrix complex polytope norms is *dense* in the set of all induced $n \times n$-matrix norms.

Theorem 4.3 *Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$. Then for any $\epsilon > 0$ there exists a b.c.p. $\mathcal{P}_\epsilon$ whose corresponding complex polytope norm $\|\cdot\|_\epsilon$ satisfies the inequalities*

$$\|x\| \le \|x\|_\epsilon \le (1+\epsilon)\|x\| \quad \text{for all } x \in \mathbb{C}^n. \tag{4.10}$$

Moreover, denoting by $\|\cdot\|$ and $\|\cdot\|_\epsilon$ also the corresponding induced matrix norms, it holds that

$$(1+\epsilon)^{-1}\|A\| \le \|A\|_\epsilon \le (1+\epsilon)\|A\| \quad \text{for all } A \in \mathbb{C}^{n\times n}. \tag{4.11}$$

5 The main results

Complex polytope norms play a particular role. In fact, Theorem 4.3 immediately implies the following refinement of Proposition 2.1.

Proposition 5.1 *The spectral radius of a bounded family $\mathcal{F}$ of complex $n \times n$-matrices is characterized by the equality*

$$\rho(\mathcal{F}) = \inf_{\|\cdot\| \in \mathcal{N}_{pol}} \|\mathcal{F}\|, \tag{5.1}$$

where $\mathcal{N}_{pol}$ denotes the set of all possible induced $n \times n$-matrix complex polytope norms.

The natural question arises whether a nondefective family admits an extremal complex polytope norm. Before trying to answer the above question we shall give an important necessary condition for the existence of an extremal complex polytope norm in Theorem 5.1. It is based on the result of Lagarias and Wang [LW95] that for finite sets of real matrices that have an extremal norm that is real *piecewise algebraic* the finiteness property holds.

Definition 5.1 *A norm $\|\cdot\|$ defined on $\mathbb{R}^k$ is said to be* piecewise algebraic *if the boundary of its unit ball is contained in the zero set of a polynomial $p(z_1, \ldots, z_k) \in \mathbb{R}[z_1, \ldots, z_k]$ such that $p(0, \ldots, 0) \neq 0$.*

Definition 5.2 *Given a bounded family $\mathcal{F}$ of complex $n \times n$-matrices, we say that the family*

$$\mathcal{F}^* = \{A^* \mid A \in \mathcal{F}\},$$

where A^* is the adjoint of the matrix A, is the adjoint family *of $\mathcal{F}$.*

The proof of the next lemma is obvious.

Lemma 5.1 *Let $\mathcal{F}$ be a bounded family of complex $n \times n$-matrices and let $\mathcal{F}^*$ its adjoint family. Then*

$$\rho(\mathcal{F}) = \rho(\mathcal{F}^*).$$

Moreover, a product $\bar{P} \in \Sigma(\mathcal{F})$ is an s.m.p. of $\mathcal{F}$ if and only if the adjoint product $\bar{P}^* \in \Sigma(\mathcal{F}^*)$ is an s.m.p. of $\mathcal{F}^*$.

The next result is a straightforward consequence of Lemma 5.1 and Corollary 4.1 with $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$.

Lemma 5.2 *A bounded nondefective family $\mathcal{F}$ of complex $n \times n$-matrices has an extremal complex polytope norm $\|\cdot\|_{\mathcal{P}}$ if and only if the adjoint family $\mathcal{F}^*$ has an extremal adjoint complex polytope norm $\|\cdot\|_{\mathcal{P}^*}$.*

Theorem 5.1 *Let $\mathcal{F} = \{A^{(i)}\}_{1 \leq i \leq m}$ be a finite nondefective family of complex $n \times n$-matrices and assume that there exists an extremal complex polytope norm $\|\cdot\|_{\mathcal{P}}$. Then the family $\mathcal{F}$ has at least an s.m.p. $\bar{P}$.*

Proof. Lemma 5.2 implies that the adjoint family $\mathcal{F}^*$ has an extremal adjoint complex polytope norm $\|\cdot\|_{\mathcal{P}^*}$. Therefore, if we show that the adjoint family $\mathcal{F}^*$ has an s.m.p. $\bar{P}^*$, then the proof is complete by virtue of Lemma 5.1. To this aim, we consider the standard vector space isomorphism Φ defined in Proposition 4.1. It naturally induces a matrix space homomorphism

$$\Phi : \mathbb{C}^{n\times n} \longrightarrow \mathbb{R}^{2n\times 2n} \quad \text{such that} \quad \Phi(A) = \begin{bmatrix} \Re(A) & -\Im(A) \\ \Im(A) & \Re(A) \end{bmatrix}.$$

It is easy to see that Φ preserves matrix products and matrix-vector products, that is

$$\Phi(AB) = \Phi(A)\Phi(B) \qquad \text{for all } A, B \in \mathbb{C}^{n\times n} \tag{5.2}$$

and

$$\Phi(Ay) = \Phi(A)\Phi(y) \qquad \text{for all } A \in \mathbb{C}^{n\times n} \text{ and } y \in \mathbb{C}^n. \tag{5.3}$$

It is also easy to see that Φ preserves convexity and that, in particular, if $\mathcal{B} \subset \mathbb{C}^n$ is the unit ball of a norm on $\mathbb{C}^n$, which is given by the Minkowski functional (see Heuser [Heu82])

$$\|y\|_{\mathcal{B}} = \inf\{r > 0 \mid y \in r\mathcal{B}\}, \tag{5.4}$$

then $\Phi(\mathcal{B}) \subset \mathbb{R}^{2n}$ is the unit ball of a norm on $\mathbb{R}^{2n}$, which is given by the Minkowski functional

$$\|z\|_{\Phi(\mathcal{B})} = \inf\{r > 0 \mid z \in r\Phi(\mathcal{B})\}. \tag{5.5}$$

In view of (5.4) and (5.5), we immediately have that

$$\|\Phi(y)\|_{\Phi(\mathcal{B})} = \|y\|_{\mathcal{B}} \qquad \text{for all } y \in \mathbb{C}^n$$

and hence that, because of (5.3), for the corresponding induced matrix norms,

$$\|\Phi(A)\|_{\Phi(\mathcal{B})} = \|A\|_{\mathcal{B}} \qquad \text{for all } A \in \mathbb{C}^{n\times n}. \tag{5.6}$$

Now consider the a.b.c.p. $\mathcal{P}^*$, that is the unit ball of the adjoint complex polytope norm $\|\cdot\|_{\mathcal{P}^*}$ such that

$$\|\mathcal{F}^*\|_{\mathcal{P}^*} = \rho(\mathcal{F}^*).$$

Since, as it is easy to see, Φ preserves matrix eigenvalues with their algebraic and geometric multiplicities, by virtue of (5.2) and (5.6) we can state that the corresponding family of real $2n \times 2n$-matrices $\Phi(\mathcal{F}^*)$ has the extremal norm $\|\cdot\|_{\Phi(\mathcal{P}^*)}$. Now, Proposition 4.1 assures that $\|\cdot\|_{\Phi(\mathcal{P}^*)}$ is a piecewise algebraic norm on $\mathbb{R}^{2n}$ (see Definition 5.1). Therefore, the proof is complete thanks to Theorem 3.2 in Lagarias and Wang [LW95]. In fact, we can claim that the family $\Phi(\mathcal{F}^*)$ has an s.m.p. $\Phi(\bar{P}^*) \in \Sigma_{k(m)}(\Phi(\mathcal{F}^*))$, where $k(m)$ depends on the cardinality m of the family $\mathcal{F}^*$. This implies that $\bar{P}^*$ is an s.m.p. for $\mathcal{F}^*$. ■

Our aim would be to reverse Theorem 5.1, that is to prove the *Complex Polytope Extremality (CPE) Theorem*. So far we were not able to find any counterexample, but not able to give a proof either. So we limit ourselves to the formulation of the following conjecture.

Conjecture 5.1 (CPE Conjecture) *Assume that a finite family of complex $n\times n$-matrices $\mathcal{F} = \{A^{(i)}\}_{1\le i\le m}$ is nondefective and has at least an s.m.p. $\bar{P}$. Then there exists an extremal complex polytope norm for $\mathcal{F}$.*

We shall be able to prove a weaker version of the above conjecture with the forthcoming *Small CPE Theorem* at the cost of some additional hypotheses on the family $\mathcal{F}$. To this aim, for any vector $x \in \mathbb{C}^n$ and for any normalized family $\hat{\mathcal{F}}$, we define the set

$$\mathcal{T}[\hat{\mathcal{F}}, x] = \{x\} \cup \{\hat{P}x \mid \hat{P} \in \Sigma(\hat{\mathcal{F}})\},$$

i.e. the *trajectory* obtained by applying all the normalized products $\hat{P}$ of matrices of $\hat{\mathcal{F}}$ to the vector x. For the convenience of the reader we recall the following characterization.

Proposition 5.2 *Let $\mathcal{F}$ be a bounded family of complex $n \times n$-matrices and let $x \in \mathbb{C}^n$. Then* $\mathrm{span}\big(\mathcal{T}[\hat{\mathcal{F}}, x]\big)$ *is the smallest linear subspace V of $\mathbb{C}^n$ containing x such that $\mathcal{F}(V) \subseteq V$.*

Proof. It is clear that, if V is a linear subspace of $\mathbb{C}^n$ containing x such that $\mathcal{F}(V) \subseteq V$, then it must necessarily contain the whole trajectory $\mathcal{T}[\hat{\mathcal{F}}, x]$. Vice versa, let $y \in \mathrm{span}\big(\mathcal{T}[\hat{\mathcal{F}}, x]\big)$. Then there exist k vectors $x^{(1)}, \ldots, x^{(k)} \in \mathcal{T}[\hat{\mathcal{F}}, x]$ and k complex numbers $\alpha_1, \ldots, \alpha_k$ (with $k \leq n$) such that $y = \sum_{i=1}^{k} \alpha_i x^{(i)}$. Now, if $A \in \mathcal{F}$, then $Ay = \sum_{i=1}^{k} \alpha_i A x^{(i)}$. Since $\rho(\mathcal{F})^{-1} A x^{(i)} \in \mathcal{T}[\hat{\mathcal{F}}, x]$, we have that $Ay \in \mathrm{span}\big(\mathcal{T}[\hat{\mathcal{F}}, x]\big)$ too. Thus the proof is complete. ■

Corollary 5.1 *Let $\mathcal{F}$ be an irreducible bounded family of complex $n \times n$-matrices and let $x \in \mathbb{C}^n$, $x \neq 0$. Then*

$$\mathrm{span}\big(\mathcal{T}[\hat{\mathcal{F}}, x]\big) = \mathbb{C}^n. \tag{5.7}$$

Proof. Since $\mathcal{F}$ is irreducible, the matrices of $\mathcal{F}$ cannot have a common nontrivial invariant linear subspace (see Definition 2.3). As a consequence, the smallest linear subspace V of $\mathbb{C}^n$ containing $x \neq 0$ such that $\mathcal{F}(V) \subseteq V$ is $\mathbb{C}^n$. ■

For a general family of matrices $\mathcal{F}$, the sets of the type

$$\mathcal{S}[\hat{\mathcal{F}}, x] = \overline{\mathrm{absco}\big(\mathcal{T}[\hat{\mathcal{F}}, x]\big)} \tag{5.8}$$

play an important role.

Proposition 5.3 *Let $\mathcal{F}$ be a nondefective bounded family of complex $n \times n$-matrices and, given a vector $x \in \mathbb{C}^n$, let (5.7) hold. Then the set $\mathcal{S}[\hat{\mathcal{F}}, x]$ is the unit ball of an extremal norm for $\mathcal{F}$.*

Proof. Since $\hat{\mathcal{F}}$ is nondefective, the trajectory $\mathcal{T}[\hat{\mathcal{F}}, x]$ is bounded for any vector $x \in \mathbb{C}^n$. Thus, because of (5.7), as in the proof of Lemma 4.1 we have that the Minkowski functional associated to $\mathcal{S} = \mathcal{S}[\hat{\mathcal{F}}, x]$ is a norm on $\mathbb{C}^n$. In order to verify that it is extremal, it is sufficient to observe that $A(\mathcal{S}) \subseteq \rho(\mathcal{F})\mathcal{S}$ for all $A \in \mathcal{F}$. ■

By virtue of the foregoing result, it appears evident that it is interesting to find conditions under which $\mathcal{S}[\hat{\mathcal{F}}, x]$ is generated by a finite number of points of the trajectory $\mathcal{T}[\hat{\mathcal{F}}, x]$. So, if (5.7) holds, the set $\mathcal{S}[\hat{\mathcal{F}}, x]$ is a b.c.p. and we can give a positive answer to the question of the existence of extremal complex polytope norms for nondefective families. In particular, in view of Corollaries 2.1 and 5.1, we can give a positive answer for irreducible families. In order to state the main result of this paper, we focus our attention on families that satisfy some particular properties.

Definition 5.3 *An eigenvector $x \neq 0$ of a matrix P related to an eigenvalue λ with $|\lambda| = \rho(P)$ is said to be a* leading eigenvector *of P.*

Definition 5.4 *Let $\mathcal{F}$ be a nondefective bounded family of complex $n \times n$-matrices. A leading eigenvector $x \neq 0$ of either an s.m.p. $\bar{P}$ of $\mathcal{F}$ or of an l.s.m.p. $\tilde{P}$ of the normalized family $\hat{\mathcal{F}}$ is said to be* leading eigenvector *of $\mathcal{F}$ (and of $\hat{\mathcal{F}}$ too).*

Remark 5.1 *Because of Theorem 2.2, any nondefective bounded family $\mathcal{F}$ has at least one leading eigenvector.*

Definition 5.5 *Let $\mathcal{F}$ be a family of complex $n \times n$-matrices. A set $\mathcal{X} \subset \mathbb{C}^n$ is said to be* $\mathcal{F}$-cyclic *if for any pair $(x, y) \in \mathcal{X} \times \mathcal{X}$ there exist $\alpha, \beta \in \mathbb{C}$ with*

$$|\alpha| \cdot |\beta| = 1 \tag{5.9}$$

and two (finite) normalized products $\hat{P}, \hat{Q} \in \Sigma(\hat{\mathcal{F}})$ such that

$$y = \alpha \hat{P} x \quad \text{and} \quad x = \beta \hat{Q} y.$$

Remark 5.2 *Because of (5.9), the normalized products $\hat{P}\hat{Q}$ and $\hat{Q}\hat{P}$ determined in the above definition are s.m.p. of the normalized family $\hat{\mathcal{F}}$ and the set $\mathcal{X}$ is necessarily included in the set $\mathcal{E}$ of the leading eigenvectors of the family $\mathcal{F}$.*

Definition 5.6 *A nondefective bounded family $\mathcal{F}$ of complex $n \times n$-matrices is said to be* asymptotically simple *if the set $\mathcal{E}$ of its leading eigenvectors is finite (modulo scalar nonzero factors) and $\mathcal{F}$-cyclic.*

As in Section 2, we shall say that a matrix Q is an *infinite power* of another matrix P if it is a limit point of the sequence $\{P^k\}_{k \geq 1}$. Observe that any eigenvalue λ of an infinite power Q of a matrix P satisfies either $|\lambda| = 1$ or $\lambda = 0$, since these are the only two possible limit values of the numeric power sequence $\{|\mu|^k\}_{k \geq 1}$ whenever $|\mu| \leq 1$. Moreover, given a nondefective matrix P with $\rho(P) = 1$, there exists at least an infinite power Q of P with an eigenvalue $\lambda = 1$, whose multiplicity is equal to the sum of the multiplicities of all the eigenvalues μ of P with $|\mu| = 1$. This easily follows from the fact that the power sequence $\{\mu^k\}_{k \geq 1}$ has the limit point 1 whenever $|\mu| = 1$ (see, for example, Hardy and Wright [HW79]).

Remark 5.3 *It follows from the above observations that, for a (nondefective) asymptotically simple family $\mathcal{F}$, each s.m.p. $\bar{P}$ and each l.s.m.p. $\tilde{P}$ have only one leading eigenvector (modulo scalar nonzero factors). Otherwise there would exist at least one l.s.m.p. of the normalized family $\hat{\mathcal{F}}$, obtained as an infinite power, with an eigenspace of dimension ≥ 2 related to the eigenvalue $\lambda = 1$. This would contradict the finiteness (modulo scalar nonzero factors) of the set $\mathcal{E}$ of leading eigenvectors.*

Observe that all the cyclic permutations of a product $\bar{P}$ have the same eigenvalues with the same multiplicities. Thus, if $\bar{P} = A^{(i_{k^*})} \ldots A^{(i_1)}$ is an s.m.p. for a family $\mathcal{F}$, then each of its cyclic permutations

$$A^{(i_s)} \ldots A^{(i_1)} A^{(i_{k^*})} \ldots A^{(i_{s+1})}, \qquad s = 1, \ldots, k^* - 1,$$

still is an s.m.p. for $\mathcal{F}$, along with all the powers of $\bar{P}$ and their cyclic permutations.

Definition 5.7 *Let $\mathcal{F}$ be a family of complex $n \times n$-matrices. An s.m.p. $\bar{P}$ is said to be* minimal *if it is not a power of another s.m.p. of $\mathcal{F}$.*

It is clear that, for any s.m.p. $\bar{P}$ of a family $\mathcal{F}$, it holds that either $\bar{P}$ is minimal or $\bar{P}$ is a power of another s.m.p., which is minimal. We have the following characterization of asymptotically simple families.

Proposition 5.4 *A nondefective bounded family $\mathcal{F}$ of complex $n \times n$-matrices is asymptotically simple if and only if it has a minimal s.m.p. $\bar{P}$ with only one leading eigenvector (modulo scalar nonzero factors) such that the set $\mathcal{E}$ of the leading eigenvectors of $\mathcal{F}$ is equal to the set of the leading eigenvectors of $\bar{P}$ and of its cyclic permutations.*

Proof. Given an s.m.p. $\bar{P}$ with only one leading eigenvector, the set of the leading eigenvectors of $\bar{P}$ and of its cyclic permutations is finite (modulo scalar nonzero factors). Indeed, it consists of k elements, k being not greater than the number of factors of $\bar{P}$. On the other hand, this set of leading eigenvectors is clearly $\mathcal{F}$-cyclic and, therefore, the sufficiency is proved. In order to prove the necessity, let $x_1, \ldots, x_s \in \mathbb{C}^n$ form a set of representatives (modulo scalar nonzero factors) of all the leading eigenvectors of $\mathcal{F}$. Since $\mathcal{F}$ is asymptotically simple, they are finitely many and, for any $i = 1, \ldots, s$, there exist $\alpha_i, \beta_i \in \mathbb{C}$ with

$$|\alpha_i| \cdot |\beta_i| = 1$$

and two (finite) normalized products $\hat{P}_i, \hat{Q}_i \in \Sigma(\hat{\mathcal{F}})$ such that

$$x_{i+1} = \alpha_i \hat{P}_i x_i \quad \text{and} \quad x_i = \beta_i \hat{Q}_i x_{i+1}$$

where it is understood that $x_{s+1} = x_1$. Therefore, we have that

$$x_1 = \alpha_1 \ldots \alpha_s \hat{P}_s \ldots \hat{P}_1 x_1 = \beta_1 \ldots \beta_s \hat{Q}_1 \ldots \hat{Q}_s x_1,$$

where

$$|\alpha_1 \ldots \alpha_s| \cdot |\beta_1 \ldots \beta_s| = 1.$$

Now, since $\rho(\hat{P}_s \ldots \hat{P}_1) \leq 1$ and $\rho(\hat{Q}_1 \ldots \hat{Q}_s) \leq 1$, it follows that

$$|\alpha_1 \ldots \alpha_s| = 1 \quad \text{and} \quad |\beta_1 \ldots \beta_s| = 1,$$

that implies

$$\rho(\hat{P}_s \ldots \hat{P}_1) = 1 \quad \text{and} \quad \rho(\hat{Q}_1 \ldots \hat{Q}_s) = 1.$$

So we can conclude that the matrix $\hat{P} = \hat{P}_s \ldots \hat{P}_1$ is an s.m.p. of $\hat{\mathcal{F}}$ such that the set of the leading eigenvectors of $\hat{P}$ and of its cyclic permutations includes (and thus is equal to) the set $\mathcal{E}$ of the leading eigenvectors of $\mathcal{F}$. The fact that x_1 is the only leading eigenvector (modulo nonzero scalar factors) of $\hat{P}$ is assured by the considerations in Remark 5.3. Finally, $\hat{P}$ can be clearly assumed to be minimal, so that is the normalization of the desired minimal s.m.p. $\bar{P}$. ■

The following definition selects a particular class of asymptotically simple families.

Definition 5.8 *A nondefective bounded family $\mathcal{F}$ of complex $n \times n$-matrices is said to be* absolutely asymptotically simple *if it is asymptotically simple and has a unique minimal s.m.p. $\bar{P}$ (modulo cyclic permutations).*

It is clear that, for absolutely asymptotically simple families, the unique minimal s.m.p. $\bar{P}$ coincides with the minimal s.m.p. given by the characterizing Proposition 5.4. Moreover, the cardinality of the set $\mathcal{E}$ of its leading eigenvectors (modulo scalar nonzero factors) is equal to the number of factors of $\bar{P}$. The result we are going to prove is the main result of this paper. It was inspired by a conjecture formulated by Maesumi [Mae95] (see also [Mae98]). Although not strictly necessary, from now on we shall consider trajectories $\mathcal{T}[\hat{\mathcal{F}}, x]$ such that condition (5.7) is satisfied. On the other hand, this is not restrictive for our purposes, since we are interested in the case that $\mathcal{S}[\hat{\mathcal{F}}, x]$ is the unit ball of a norm.

Theorem 5.2 (Small CPE Theorem) *Assume that a finite family $\mathcal{F} = \{A^{(i)}\}_{1 \le i \le m}$ of complex $n \times n$-matrices is nondefective and asymptotically simple. Moreover, let $x \neq 0$ be a leading eigenvector of $\mathcal{F}$ and assume that (5.7) is satisfied. Then the set*

$$\partial \mathcal{S}[\hat{\mathcal{F}}, x] \bigcap \mathcal{T}[\hat{\mathcal{F}}, x] \tag{5.10}$$

is finite modulo scalar factors of unitary modulus. As a consequence, there exist a finite number of normalized products $\hat{P}^{(1)}, \ldots, \hat{P}^{(s)} \in \Sigma(\hat{\mathcal{F}})$ such that

$$\mathcal{S}[\hat{\mathcal{F}}, x] = \text{absco}\Big(\{x, \hat{P}^{(1)}x, \ldots, \hat{P}^{(s)}x\}\Big), \tag{5.11}$$

so that $\mathcal{S}[\hat{\mathcal{F}}, x]$ is a b.c.p.

Before giving the proof of this theorem, we state a useful technical lemma.

Lemma 5.3 *Let $\mathcal{F} = \{A^{(i)}\}_{1 \le i \le m}$ be a nondefective finite family of complex $n \times n$-matrices and, given a vector $x \in \mathbb{C}^n$, assume that (5.7) is satisfied and that the (bounded) set $\partial \mathcal{S}[\hat{\mathcal{F}}, x] \bigcap \mathcal{T}[\hat{\mathcal{F}}, x]$, modulo scalar factors of unitary modulus, is not finite. Then there exists a sequence of distinct vectors*

$$x^{(k)} \in \partial \mathcal{S}[\hat{\mathcal{F}}, x] \bigcap \mathcal{T}[\hat{\mathcal{F}}, x]$$

with $x^{(1)} = x$ such that, for all $k \ge 1$,

$$x^{(k+1)} = \hat{A}^{(\ell_k)} x^{(k)} \quad \text{for some } \ell_k \in \{1, \ldots, m\},$$

where $\hat{A}^{(i)} = A^{(i)}/\rho(\mathcal{F}) \in \hat{\mathcal{F}}$, $1 \le i \le m$, and such that, whenever $k \neq h$,

$$x^{(k)} \neq u x^{(h)} \quad \text{for all } u \in \mathbb{C} \text{ with } |u| = 1.$$

Proof. We prove the result by induction. Define $x^{(1)} = x$. By hypothesis, there exist infinitely many distinct vectors of the type $\hat{Q}x^{(1)}$, no two of which are multiples of one another by a scalar factor of unitary modulus, with $\hat{Q} \in \Sigma(\hat{\mathcal{F}})$ and $\hat{Q}x^{(1)} \in \partial \mathcal{S}[\hat{\mathcal{F}}, x]$. Since $\hat{\mathcal{F}}$ is finite, there exists a matrix $\hat{A}^{(\ell_1)}$ for which, with $x^{(2)} = \hat{A}^{(\ell_1)} x^{(1)}$, it holds that $x^{(2)} \neq u x^{(1)}$ for all $u \in \mathbb{C}$ with $|u| = 1$. Moreover, $x^{(2)}$ can be chosen so that infinitely many distinct vectors of the type $\hat{Q}x^{(2)}$, not proportional to one another by scalar factors of unitary modulus, exist such that $\hat{Q} \in \Sigma(\hat{\mathcal{F}})$ and $\hat{Q}x^{(2)} \in \partial \mathcal{S}[\hat{\mathcal{F}}, x]$. Since (5.7) holds, by Propositions 5.3 and 3.2 we immediately have that $x^{(2)} \in \partial \mathcal{S}[\hat{\mathcal{F}}, x]$. Now assume that there exist $k \ge 2$ distinct vectors $x^{(1)} \neq x, x^{(2)}, \ldots, x^{(k)} \in \partial \mathcal{S}[\hat{\mathcal{F}}, x]$, such that $x^{(i)} \neq u x^{(j)}$ for all $u \in \mathbb{C}$ with $|u| = 1$ if $i \neq j$, such that $x^{(j)} = \hat{A}^{(\ell_{j-1})} x^{(j-1)}$ for some $\ell_{j-1} \in \{1, \ldots, m\}$ and for which infinitely many distinct vectors of the type $\hat{Q}x^{(j)}$, not proportional to one another by scalar factors of unitary modulus, exist with $\hat{Q} \in \Sigma(\hat{\mathcal{F}})$ and $\hat{Q}x^{(j)} \in \Sigma(\hat{\mathcal{F}})$. Since $\hat{\mathcal{F}}$ is finite, there exists another matrix $\hat{A}^{(\ell_k)}$ for which, with $x^{(k+1)} = \hat{A}^{(\ell_k)}x^{(k)}$, it holds that $x^{(k+1)} \neq ux^{(j)}$, for all $u \in \mathbb{C}$ with $|u| = 1$, $j = 1, \ldots, k$, and infinitely many distinct vectors of the type $\hat{Q}x^{(k+1)}$, not proportional to one another by scalar factors of unitary modulus, such that $\hat{Q} \in \Sigma(\hat{\mathcal{F}})$ and $\hat{Q}x^{(k+1)} \in \partial\mathcal{S}[\hat{\mathcal{F}}, x]$. Again, by Propositions 5.3 and 3.2, it turns out that $x^{(k+1)} \in \partial\mathcal{S}[\hat{\mathcal{F}}, x]$. This completes the induction step and the result is proved. ■

Proof of Theorem 5.2. Denoting by $\mathcal{E}$ the set of the leading eigenvectors of the family $\mathcal{F}$, let us consider

$$\Xi = \mathcal{E} \bigcap \partial\mathcal{S}[\hat{\mathcal{F}}, x].$$

Since the family $\mathcal{F}$ is asymptotically simple and $\mathcal{S}[\hat{\mathcal{F}}, x]$ is the unit ball of a norm, the set Ξ is finite modulo scalar factors of unitary modulus. On the contrary, by contradiction, assume that the set

$$\partial\mathcal{S}[\hat{\mathcal{F}}, x] \bigcap \mathcal{T}[\hat{\mathcal{F}}, x],$$

even if considered modulo scalar factors of unitary modulus, is not finite, so that can be applied Lemma 5.3 to obtain the sequence $\{x^{(k)}\}_{k\geq 1}$ with $x^{(1)} = x$. Therefore, since $x^{(1)} \in \Xi \bigcap \mathcal{T}[\hat{\mathcal{F}}, x]$, there exists $j \geq 1$ such that

$$x^{(j+1)} \notin \Xi \quad \text{and} \quad x^{(i)} \in \Xi \bigcap \mathcal{T}[\hat{\mathcal{F}}, x] \quad \text{for all } i \leq j. \tag{5.12}$$

Since $\Sigma(\hat{\mathcal{F}})$ is bounded, the resulting sequence of normalized matrix products $\hat{B}^{(k)} = \hat{A}^{(\ell_{k-1})} \ldots \hat{A}^{(\ell_{j+1})}$ such that $x^{(k)} = \hat{B}^{(k)}x^{(j+1)}$ has a subsequence $\{\hat{B}^{(k_s)}\}_{s\geq 1}$ that converges to a limit point $\hat{B}$ in $\overline{\Sigma(\hat{\mathcal{F}})}$. Therefore, also the subsequence of vectors $\{x^{(k_s)}\}_{s\geq 1}$ has a limit point $y = \hat{B}x^{(j+1)}$. Moreover, since $\partial\mathcal{S}[\hat{\mathcal{F}}, x]$ is closed, we have that

$$\hat{B}x^{(j+1)} \in \partial\mathcal{S}[\hat{\mathcal{F}}, x]. \tag{5.13}$$

For each $s \geq 1$ there exists a matrix $\hat{R}^{(s)} \in \Sigma(\hat{\mathcal{F}})$ such that

$$\hat{B}^{(k_{s+1})} = \hat{R}^{(s)}\hat{B}^{(k_s)}.$$

Again for the boundedness of $\Sigma(\hat{\mathcal{F}})$, the sequence $\{\hat{R}^{(s)}\}_{s\geq 1}$ has a limit point $\hat{R}$ in $\overline{\Sigma(\hat{\mathcal{F}})}$. By passing to the limit, this allows us to conclude that

$$\hat{B}x^{(j+1)} = \hat{R}\hat{B}x^{(j+1)}.$$

In other words, $\hat{R}$ is an l.s.m.p. of $\hat{\mathcal{F}}$ and

$$\hat{B}x^{(j+1)} \in \Xi.$$

Since $\mathcal{F}$ is asymptotically simple, the set Ξ is $\mathcal{F}$-cyclic and, hence, there exist $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \cdot |\beta| = 1$ and two (finite) normalized products $\hat{P}, \hat{Q} \in \Sigma(\hat{\mathcal{F}})$ such that

$$\hat{B}x^{(j+1)} = \alpha\hat{P}x^{(j)} \quad \text{and} \quad x^{(j)} = \beta\hat{Q}\hat{B}x^{(j+1)}. \tag{5.14}$$

This implies

$$\hat{Q}\hat{P}x^{(j)} = (\alpha\beta)^{-1}x^{(j)}$$

and thus, since $|(\alpha\beta)^{-1}| = 1$, by Proposition 5.3 and Theorem 3.1 we have that

$$\hat{P}x^{(j)} \in \partial\mathcal{S}[\hat{\mathcal{F}}, x].$$

Therefore, by (5.13) and (5.14), it turns out that

$$|\alpha| = |\beta| = 1. \tag{5.15}$$

In conclusion, we have

$$\beta^{-1}x^{(j+1)} = \beta^{-1}\hat{A}^{(\ell_j)}x^{(j)} = \hat{A}^{(\ell_j)}\hat{Q}\hat{B}x^{(j+1)}.$$

Now, since $\hat{A}^{(\ell_j)}\hat{Q}\hat{B} \in \overline{\Sigma(\hat{\mathcal{F}})}$ and (5.15) holds, we can state that $x^{(j+1)}$ is a leading eigenvector of $\mathcal{F}$, i.e.

$$x^{(j+1)} \in \Xi,$$

in contradiction with (5.12). ■

Remark that, if all the matrices of the family $\mathcal{F}$ are real and if also the starting leading eigenvector x is real, then Theorem 5.2 determines a classical polytope in $\mathbb{R}^n$. The next results are useful for a deeper understanding of the structure of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}}, x]$ obtained under the hypotheses of Theorem 5.2.

Theorem 5.3 *Let the hypotheses of Theorem 5.2 hold. Then each leading eigenvector ξ of $\mathcal{F}$ in the set $\Xi = \mathcal{E} \bigcap \partial\mathcal{S}[\hat{\mathcal{F}}, x]$ satisfies one of the following two statements:*

(a) *ξ is a vertex of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}}, x]$;*

(b) *there exist $s \geq 2$ vertices $\xi_1, \ldots, \xi_s$ of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}}, x]$ such that*

$$\xi_1, \ldots, \xi_s \in \Xi \quad \text{and} \quad \xi \in \operatorname{absco}\bigl(\{\xi_1, \ldots, \xi_s\}\bigr). \tag{5.16}$$

Proof. Consider a leading eigenvector $\xi \in \Xi$ and assume that it is not a vertex of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}}, x]$. Then there must exist $s \geq 2$ vertices $\xi_1, \ldots, \xi_s$ of $\mathcal{S}[\hat{\mathcal{F}}, x]$ such that

$$\xi = \sum_{i=1}^{s} \lambda_i \xi_i$$

with

$$\lambda_i \neq 0,\ i = 1, \ldots, s, \quad \text{and} \quad \sum_{i=1}^{s} |\lambda_i| = 1. \tag{5.17}$$

We are left to prove that $\xi_1, \ldots, \xi_s \in \Xi$.

Since ξ is a leading eigenvector of $\mathcal{F}$, there exists an l.s.m.p. (possibly an s.m.p.) $\tilde{P}$ of $\hat{\mathcal{F}}$ such that $\tilde{P}\xi = u\xi$ with $|u| = 1$. Thus, denoting by $\|\cdot\|$ the complex polytope norm determined by $\mathcal{S}[\hat{\mathcal{F}}, x]$, for any $k \geq 1$ we have

$$1 = \|\xi\| = \|\tilde{P}^k\xi\| \leq \sum_{i=1}^{s} |\lambda_i| \cdot \|\tilde{P}^k\xi_i\|.$$

Since $\|\tilde{P}^k\xi_i\| \leq 1$, in view of (5.17) we can claim that $\|\tilde{P}^k\xi_i\| = 1$, that is

$$\tilde{P}^k\xi_i \in \partial\mathcal{S}[\hat{\mathcal{F}}, x], \quad i = 1, \ldots, s.$$

Denoting by $\tilde{P}^\infty$ an infinite power of $\tilde{P}$, in the same way as we did in the proof of Theorem 5.2, we can prove that

$$\tilde{P}^\infty \xi_i \in \Xi, \quad i = 1, \ldots, s.$$

On the other hand, since the set Ξ is $\mathcal{F}$-cyclic and included in $\partial\mathcal{S}[\hat{\mathcal{F}}, x]$, again in the same way as in the proof of Theorem 5.2, we can prove that there exist (finite) normalized products $\hat{Q}, \hat{R}_i \in \Sigma(\hat{\mathcal{F}})$ and $v, w_i \in \mathbb{C}$ with $|v| = |w_i| = 1$ such that

$$x = v\hat{Q}\xi \quad \text{and} \quad \xi = w_i \hat{R}_i \tilde{P}^\infty \xi_i, \; i = 1, \ldots, s.$$

Moreover, since all the vertices of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}}, x]$ obviously belong to $\mathcal{T}[\hat{\mathcal{F}}, x]$ (modulo scalar factors of unitary modulus), there exist (finite) normalized products $\hat{S}_i \in \Sigma(\hat{\mathcal{F}})$ such that

$$\xi_i = \hat{S}_i x, \quad i = 1, \ldots, s.$$

In conclusion, we have

$$\xi_i = v w_i \hat{S}_i \hat{Q} \hat{R}_i \tilde{P}^\infty \xi_i, \quad i = 1, \ldots, s,$$

where $|vw_i| = 1$, that implies

$$\xi_i \in \Xi, \quad i = 1, \ldots, s.$$

■

Corollary 5.2 *Let the hypotheses of Theorem 5.2 hold and, moreover, let the family $\mathcal{F}$ be absolutely asymptotically simple. Then all the leading eigenvectors of $\mathcal{F}$ (in the set $\Xi = \mathcal{E} \bigcap \partial\mathcal{S}[\hat{\mathcal{F}}, x]$) are vertices of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}}, x]$.*

Proof. Assume, by contradiction, that there exists a leading eigenvector $\xi \in \Xi$ which is not a vertex of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}}, x]$. Then it necessarily satisfies statement (b) of Theorem 5.3.

On the other hand, there exists a unique normalized minimal s.m.p. $\hat{P}$ such that $\hat{P}\xi = u\xi$ with $|u| = 1$. Therefore, for each ξ_i appearing in statement (b) there exists a proper normalized right factor $\hat{P}_i$ of the s.m.p. $\hat{P}$ such that $\xi_i = u_i \hat{P}_i \xi$ with $|u_i| = 1$.

Thus we obtain

$$\xi_i \in \mathrm{absco}\Big(\{\hat{P}_i \xi_1, \ldots, \hat{P}_i \xi_s\}\Big).$$

Now, since the essential system of vertices of a b.c.p. is unique modulo scalar factors of unitary modulus (see Proposition 4.2) and since ξ_i is a vertex of $\mathcal{S}[\hat{\mathcal{F}}, x]$, it necessarily holds that, for all $j = 1, \ldots, s$,

$$\xi_i = v_j \hat{P}_i \xi_j \quad \text{with } |v_j| = 1.$$

In particular, we have that

$$\xi_i = v_i \hat{P}_i \xi_i \quad \text{with } |v_i| = 1,$$

that is the proper normalized right factor $\hat{P}_i$ is an s.m.p., against the uniqueness of $\hat{P}$ (modulo cyclic permutations). ■

6 On the necessity of the assumptions in the Small CPE Theorem

The assumptions of Theorem 5.2 may seem to be somewhat restrictive. On the other hand, now we present some simple examples that imply their necessity for the finiteness (modulo scalar factors of unitary modulus) of the set $\partial\mathcal{S}[\hat{\mathcal{F}}, x] \bigcap \mathcal{T}[\hat{\mathcal{F}}, x]$ in the general case. It is just the case to remark that, for the actual construction of the unit ball of an extremal complex polytope norm by means of some suitable algorithm, the finiteness property mentioned above is strongly recommendable. Indeed, one of the aims of our future work will be the improvement of the currently available algorithms for the computation and approximation of the spectral radius of a family of matrices by using the theoretical results developed in the present paper. In the following two examples the considered families are not asymptotically simple. More precisely, in Example 6.1 the set of leading eigenvectors is not finite (modulo scalar nonzero factors) and not $\mathcal{F}$-cyclic either. In Example 6.2 it is finite but not $\mathcal{F}$-cyclic.

Example 6.1 Consider the 2×2-matrix family $\mathcal{F} = \{A, B\}$, where

$$A = \begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \frac{1}{2} & -\frac{\mathrm{i}}{2} \\ 0 & 0 \end{bmatrix}.$$

The eigenvalues of A are e^{i} and $\mathrm{e}^{-\mathrm{i}}$ with corresponding eigenvectors $[1, -\mathrm{i}]^T$ and $[1, \mathrm{i}]^T$, respectively. The eigenvalues of B are $\frac{1}{2}$ and 0. It is easy to see that $\rho(\mathcal{F}) = 1$, that A is the unique minimal s.m.p. of $\mathcal{F}$ and that all the l.s.m.p. of $\mathcal{F}$ are infinite powers of A. Since the s.m.p. A has two leading eigenvectors, according to Remark 5.3, the identity matrix I is an l.s.m.p., so that all the vectors of $\mathbb{C}^2$ are leading eigenvectors of $\mathcal{F}$.

It is immediately seen that the family $\mathcal{F}$ is not even $\mathcal{F}$-cyclic. As a consequence of the violation of the hypotheses of Theorem 5.2, now we are going to see that, by setting

$$x = [1, \mathrm{i}]^T,$$

which is a leading eigenvector of the s.m.p. A, the set $\mathcal{S}[\mathcal{F}, x]$ is not a b.c.p. In fact, it holds that

$$Bx = [1, 0]^T \qquad \text{and} \qquad A^k Bx = [\cos(k), \sin(k)]^T, \quad k \geq 1.$$

All these vectors lie on the unit circle $\mathcal{C}$ of $\mathbb{R}^2$ and form a set which is dense in $\mathcal{C}$. On the other hand,

$$BA^k Bx = \frac{1}{2}\mathrm{e}^{-\mathrm{i}k} Bx,$$

and, hence, we can conclude that

$$\mathcal{S}[\mathcal{F}, x] = \mathrm{absco}\{x, \mathcal{C}\},$$

that is not a b.c.p. Finally, note that the infinitely many vectors of the trajectory $\mathcal{T}[\mathcal{F}, x]$, namely $A^k Bx$ for $k \geq 0$, which are not proportional to one another, lie on the boundary $\mathcal{C}$ of $\mathcal{S}[\mathcal{F}, x]$. $\diamond$

Example 6.2 Consider the 2×2-matrix family $\mathcal{F} = \{A, B\}$, where

$$A = \begin{bmatrix} 3 - \sqrt{5} & \frac{3-\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} & 3 - \sqrt{5} \end{bmatrix}.$$

The matrices A and B have the same eigenvalues 1 and $\frac{7-3\sqrt{5}}{2}$. It can be seen that $\rho(\mathcal{F}) = 1$, that A and B are the only minimal s.m.p. of $\mathcal{F}$ and that the unique l.s.m.p. of $\mathcal{F}$ are just $A^\infty = \lim_{k\to\infty} A^k$ and $B^\infty = \lim_{k\to\infty} B^k$. Therefore, the set of the leading eigenvectors of $\mathcal{F}$ is given by the leading eigenvectors of A and B, namely $\alpha[\frac{1+\sqrt{5}}{2}, 1]^T$, $\alpha \neq 0$, and $\alpha[1, \frac{1+\sqrt{5}}{2}]^T$, $\alpha \neq 0$, respectively. This set is finite (modulo scalar nonzero factors) but, as it is easy to see, it is not $\mathcal{F}$-cyclic. As in the previous example, the hypotheses of Theorem 5.2 are violated, even if to a smaller extent. Now set

$$x = \left[\frac{1+\sqrt{5}}{2}, 1\right]^T,$$

which is the leading eigenvector of the s.m.p. A. It can be checked that all the vectors $B^k x$, $k \geq 1$, are distinct and lie on the segment of $\mathbb{R}^2$ that joins x and

$$B^\infty x = \frac{14+6\sqrt{5}}{15+7\sqrt{5}} \left[1, \frac{1+\sqrt{5}}{2}\right]^T,$$

which is the leading eigenvector of the s.m.p. B, and that all the vectors $AB^k x$, $k \geq 1$, lie inside the b.c.p. $\text{absco}\{x, B^\infty x\}$. We can conclude that

$$\mathcal{S}[\mathcal{F}, x] = \text{absco}\{x, B^\infty x\}.$$

Although $\mathcal{S}[\mathcal{F}, x]$ is a b.c.p., we have seen that the finiteness property of the set $\partial\mathcal{S}[\hat{\mathcal{F}}, x] \bigcap \mathcal{T}[\hat{\mathcal{F}}, x]$ assured by Theorem 5.2 does not hold. $\diamond$

In the next example the considered family is asymptotically simple (even absolutely). Nevertheless, we shall see the substantial potential difference between a trajectory $\mathcal{T}[\mathcal{F}, x]$ exiting from a leading eigenvector and from a vector x which is not such.

Example 6.3 Consider the 3×3-matrix family $\mathcal{F} = \{A, B\}$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

Since both A and B are lower triangular, it is immediately seen that $\rho(\mathcal{F}) = 1$ and that A is the unique minimal s.m.p., which has only one leading eigenvector (modulo scalar non zero factors), related to the eigenvalue $\lambda = 1$, given by $\alpha[1, 0, 0]^T$, $\alpha \neq 0$. Therefore, $\mathcal{F}$ is absolutely asymptotically simple. In view of Theorem 5.2, if we choose

$$x = [1, 0, 0]^T,$$

then the corresponding set $\mathcal{S}[\mathcal{F}, x]$ is a b.c.p. generated by a finite number of points of the trajectory $\mathcal{T}[\mathcal{F}, x]$. In order to verify this fact, we observe that

$$Bx = \frac{1}{2}[1, 1, 1]^T, \quad B[0, 1, 0]^T = B[0, 0, 1]^T = [0, 0, 0]^T,$$

$$A^k[1, 1, 1]^T = \left[1, \frac{1}{2^k}, \frac{1}{4^k}\right]^T, \quad k \geq 1.$$

Consequently, it easily turns out that

$$\mathcal{S}[\mathcal{F},x] = \mathrm{absco}\{x, Bx, ABx, A^2Bx\}$$

and that all the other vectors of the trajectory $\mathcal{T}[\mathcal{F},x]$ lie inside the above b.c.p. Now choose instead

$$x = [1,1,1]^T,$$

which is not a leading eigenvector. This time $\mathcal{S}[\mathcal{F},x]$ is not a b.c.p. In fact, the infinitely many vectors $A^k x$, $k \geq 1$, which all belong to $\partial\mathcal{S}[\hat{\mathcal{F}},x]$ and accumulate at $[1,0,0]^T$, are such that their projections into the plane (x_2,x_3) of the last two variables lie on the parabola whose equation is $x_3 = x_2^2$. $\diamond$

We remark that, although necessary for the general validity of the finiteness property of the set $\partial\mathcal{S}[\hat{\mathcal{F}},x] \bigcap \mathcal{T}[\hat{\mathcal{F}},x]$, the special hypotheses of Theorem 5.2 seem not to be necessary for the simple existence of an extremal complex polytope (see the CPE Conjecture). Indeed, even for the family of Example 6.1, it can be easily verified that a b.c.p. unit ball is given by

$$\mathcal{P} = \mathrm{absco}\{[1,\mathrm{i}]^T, [1,-\mathrm{i}]^T\}.$$

We conclude the paper with the following example which shows that, for a family which is asymptotically simple but not absolutely, not all the leading eigenvectors are necessarily vertices of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}},x]$.

Example 6.4 Consider the 2×2-matrix family $\mathcal{F} = \{A,B,C,D\}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad D = \begin{bmatrix} \frac{2}{3} & 0 \\ -\frac{2}{3} & 0 \end{bmatrix}.$$

It turns out that $XY = X$ and $XD = O$, the zero matrix, for all $X,Y \in \{A,B,C\}$. Therefore, since $\rho(A) = \rho(B) = \rho(C) = 1$ and $\rho(D) = \frac{2}{3}$, we have that $\rho(\mathcal{F}) = 1$ and that A,B,C are all minimal s.m.p. of $\mathcal{F}$. Moreover, the leading eigenvectors of $\mathcal{F}$ are just the three leading eigenvectors of A,B and C, that is $\alpha[1,0]^T$, $\beta[0,1]^T$ and $\gamma[\frac{1}{2},\frac{1}{2}]^T$, $\alpha,\beta,\gamma \neq 0$, respectively, which are related to the common eigenvalue $\lambda = 1$. This set of leading eigenvectors is clearly finite and, as is immediately seen, also $\mathcal{F}$-cyclic. Therefore, $\mathcal{F}$ is asymptotically simple, but not absolutely. If we choose

$$x = [1,0]^T,$$

which is the leading eigenvector of A, it easily turns out that

$$\mathcal{S}[\mathcal{F},x] = \mathrm{absco}\{x, Bx, Dx\},$$

where $Bx = [0,1]^T$ is the leading eigenvector of B and $Dx = [\frac{2}{3},-\frac{2}{3}]^T$ is not a leading eigenvector of $\mathcal{F}$. Remark that the leading eigenvector of C belonging to $\partial\mathcal{S}[\hat{\mathcal{F}},x]$, namely $Cx = [\frac{1}{2},\frac{1}{2}]^T$, satisfies the convexity relationship $Cx = \frac{1}{2}x + \frac{1}{2}Bx$ and, hence, is not a vertex of the b.c.p. $\mathcal{S}[\mathcal{F},x]$. $\diamond$

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